Signals and Systems-23EC02

UNIT-V

Laplace Transforms and Z-Transforms

B.Tech., III-Sem., ECE

1. **Concept of Laplace Transform**
2. **Relation between Laplace and Fourier Transform**
3. **Existence of Laplace Transform**
4. **Laplace Transform of Various Classes of Signals**
5. **Region of Convergence (ROC) in s-domain and Properties**
6. **Properties of Laplace Transform**
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9. **Z-Transform of various classes of Signals**
10. **Region of Convergence (ROC) in z-domain and Properties**
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1. Concept of Laplace Transform:

Laplace Transform is a mathematical tool, which is used to evaluate the frequency domain(s-domain) representation of a given continuous time domain signal.

Laplace Transform of a continuous time signal x(t) is represented with X(s) and it can be obtained from the formula

$$\text{LT}[\,x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt ------(1)$$

Where, s is a complex variable, s = σ + jw

σ = Re{s} = Real part of s

w = Im{s} = Imaginary part of s

If x(t) is right sided (causal), then its s-domain can be obtained from the formula

$$\text{LT}[\,x(t)\,] = X(s) = \int_{0}^{\infty} x(t)e^{-st}dt ----- (2)$$

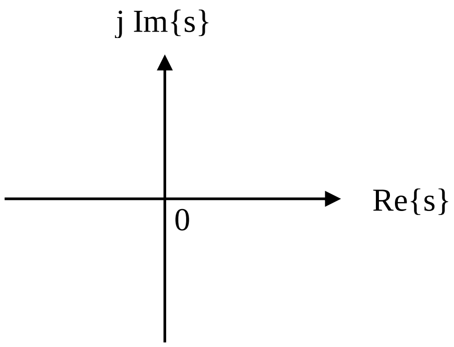
If x(t) is left sided (anti-causal), then its s-domain can be obtained from the formula

$$\text{LT}[\,x(t)\,] = X(s) = \int_{-\infty}^{0} x(t)e^{-st}dt ----- (3)$$

- Equation (1) is called bilateral Laplace Transform
- Equations (2) and (3) are called unilateral Laplace Transforms

S-Plane and Pole-Zero Plot:

A graph, which is drawn between Re{s} on x-axes and jIm{s} on y-axes is called s-plane.



Let, $X(s)=\dfrac{(s-z_1)(s-z_2)(s-z_3)\ldots..}{(s-p_1)(s-p_2)(s-p_3)\ldots..}$

- Roots of numerator polynomial are called zeros and which are represented with ‘o’.
- Roots of denominator polynomial are called poles and which are represented with ‘x’.
- Indicate poles and zeros on s-plane to get pole-zero plot.

2. Relation between Laplace and Fourier Transforms:

From the basic definition of Fourier Transform

$$\text{FT}[\,x(t)\,] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

From the basic definition of Laplace Transform

$$\text{LT}[\,x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt\,; put\ s = \sigma + jw$$

$$= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+jw)t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-jwt}dt$$

$$= FT[x(t)e^{-\sigma t}]$$

If $\sigma = 0 \Rightarrow s = jw$, then LT [x(t)] = FT [x(t)].

On the imaginary axes of s-plane, both the Laplace and Fourier Transforms are same.

3. Existance of Laplace Transdorm:

The product of given signal x(t) and the exponential term e^{-st} should be absolutely integrable is called existence of Laplace Transform or convergence of Laplace Transform.

$$\int_{-\infty}^{\infty} |x(t)e^{-st}|dt < \infty; s = \sigma + jw$$

$$\Rightarrow \int_{-\infty}^{\infty} \left|x(t)e^{-(\sigma+jw)t}\right|dt < \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}e^{-jwt}\right|dt < \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} |x(t)||e^{-\sigma t}|\left|e^{-jwt}\right|dt < \infty$$

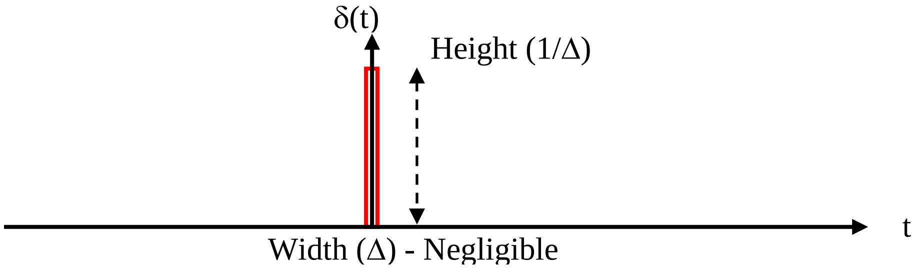
$$\Rightarrow \int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}dt < \infty$$

Note: The range of values of 'σ' or 'Re{s}' or 's' for which the basic definition of Laplace Transform will converges or produces a finite result is called Region of Convergence (ROC).

4. Laplace Transform of various classes of Signals:

4.1. Impulse Signal, x(t) = δ(t):

$$Impulse\ signal, x(t) = \delta(t) = \begin{cases} \infty & ; \ t = 0 \\ 0 & ; \ t \neq 0 \end{cases}$$



From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\Rightarrow LT[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt\,; Property\ of\ impulse\ signal, \delta(t)x(t) = \delta(t)x(0)$$

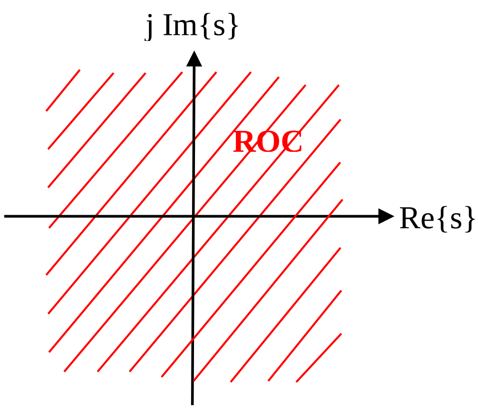
$$= \int_{-\infty}^{\infty} \delta(t)e^{0}dt$$

$$= \int_{-\infty}^{\infty} \delta(t)1dt$$

$$= \int_{-\infty}^{\infty} \delta(t)dt\,; Area\ under\ impulse\ signal\ is\ '1'$$

$$= 1$$

$LT[\delta(t)] = X(s) = 1$	ROC
	$Entire\ s - plaane$



4.2. Step Signal, $x(t) = u(t)$:

From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\Rightarrow LT[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-st}dt, u(t) = 1, t > 0$$

$$= \int_{0}^{\infty} e^{-st}dt$$

$$= \frac{e^{-st}}{-s}\Big|_{0}^{\infty}$$

$$= \frac{e^{-s\infty} - e^{s0}}{-s}$$

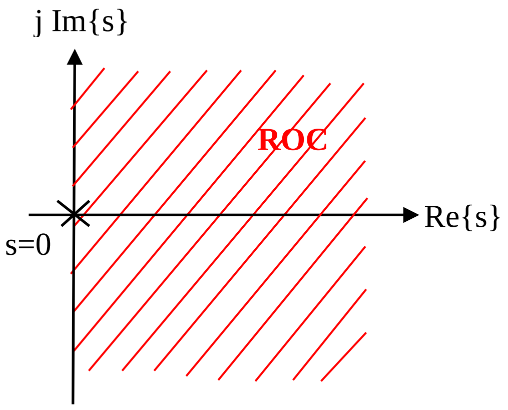
$$= \frac{e^{-\infty} - e^{0}}{-s}, s > 0$$

$$= \frac{0 - 1}{-s}, s > 0$$

$$= \frac{1}{s}, s > 0$$

$LT[u(t)] = X(s) = \frac{1}{s}$	ROC
	$s > 0$

Pole-Zero Plot with ROC:



Note: X(s) has one pole, which is located at s = 0.

4.3. Decaying Exponential Signal, $x(t) = e^{-at}u(t), a > 0$:

From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$LT[e^{-at}u(t)] = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt, u(t) = 1, t > 0$$

$$= \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$= \frac{e^{-(s+a)t}}{-(s+a)}\Bigg|_{0}^{\infty}$$

$$= \frac{e^{-(s+a)\infty} - e^{(s+a)0}}{-(s+a)}$$

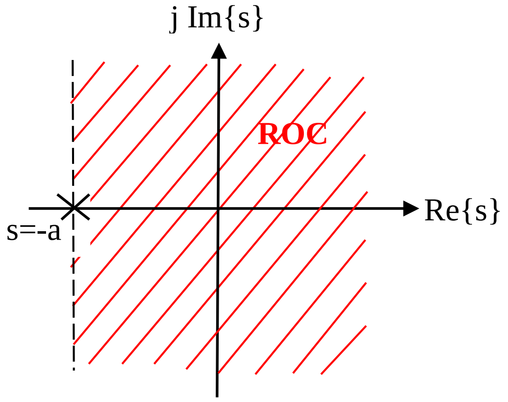
$$= \frac{e^{-\infty} - e^{0}}{-(s+a)}, s + a > 0$$

$$= \frac{0-1}{-(s+a)}, s > -a$$

$$= \frac{1}{s+a}, s > -a$$

	ROC
$LT[e^{-at}u(t)] = X(s) = \dfrac{1}{s+a}$	$s > -a$

Pole-Zero Plot with ROC:



Note: X(s) has one pole, which is located at s = -a.

4.4. Raising Exponential Signal, $x(t) = e^{at}u(-t), a > 0$:

From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$LT[e^{at}u(-t)] = \int_{-\infty}^{\infty} e^{at}u(-t)e^{-st}dt;\ u(-t) = 1, t < 0$$

$$= \int_{-\infty}^{0} e^{(a-s)t}dt$$

$$= \frac{e^{(a-s)t}}{a-s}\Bigg|_{-\infty}^{0}$$

$$= \frac{e^{(a-s)0} - e^{(a-s)(-\infty)}}{a-s}$$

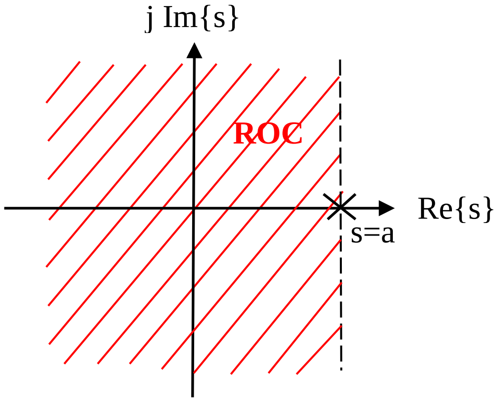
$$= \frac{e^{0} - e^{-\infty}}{a-s}, a - s > 0$$

$$= \frac{1-0}{a-s}, a > s$$

$$= \frac{-1}{s-a}, s < a$$

$LT[e^{at}u(-t)] = X(s) = \dfrac{-1}{s-a}$	ROC
	$s < a$

Pole-Zero Plot with ROC:



Note: X(s) has one pole, which is located at s = a.

4.5. Signal, $x(t) = -e^{-at}u(-t), a > 0$:

From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\Rightarrow LT[-e^{-at}u(-t)] = \int_{-\infty}^{\infty} \left(-e^{-at}u(-t)\right)e^{-st}dt;\ u(-t) = 1, t < 0$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

$$= -\frac{e^{-(s+a)t}}{-(s+a)}\Bigg|_{-\infty}^{0}$$

$$= \frac{e^{-(s+a)0} - e^{-(s+a)(-\infty)}}{s+a}$$

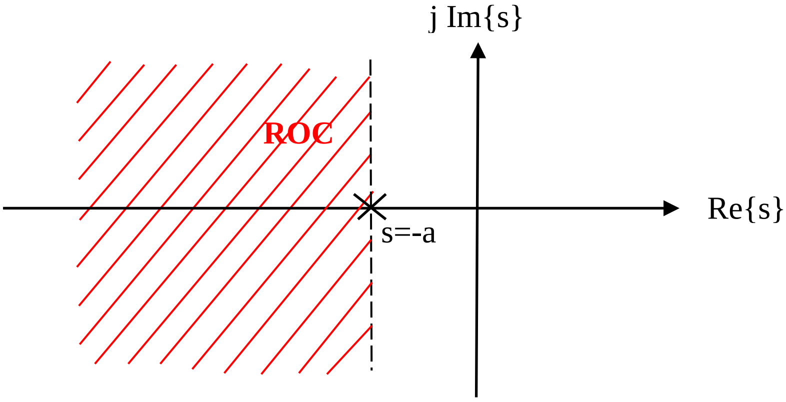
$$= \frac{e^{0} - e^{-\infty}}{s+a}, s+a < 0$$

$$= \frac{1-0}{s+a}, s < -a$$

$$= \frac{1}{s+a}, s < -a$$

$LT[-e^{-at}u(-t)] = X(s) = \dfrac{1}{s+a}$	ROC
	$s < -a$

Pole-Zero Plot with ROC:



Note: X(s) has one pole, which is located at s = -a.

4.6. Double Exponential Signal, $x(t) = e^{-a|t|}, a > 0$:

From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\Rightarrow LT[e^{-a|t|}] = \int_{-\infty}^{\infty} e^{-a|t|}e^{-st}dt$$

$$= \int_{-\infty}^{0} e^{-a(-t)}e^{-st}dt + \int_{0}^{\infty} e^{-a(t)}e^{-st}dt$$

$$= \int_{-\infty}^{0} e^{at}e^{-st}dt + \int_{0}^{\infty} e^{-at}e^{-st}dt$$

$$= \int_{-\infty}^{0} e^{(a-s)t}dt + \int_{0}^{\infty} e^{-(a+s)t}dt$$

$$= \frac{e^{(a-s)t}}{a-s}\Bigg|_{-\infty}^{0} + \frac{e^{-(a+s)t}}{-(a+s)}\Bigg|_{0}^{\infty}$$

$$= \frac{e^{(a-s)0} - e^{(a-s)(-\infty)}}{a-s} + \frac{e^{-(a+s)\infty} - e^{-(a+s)0}}{-(a+s)}$$

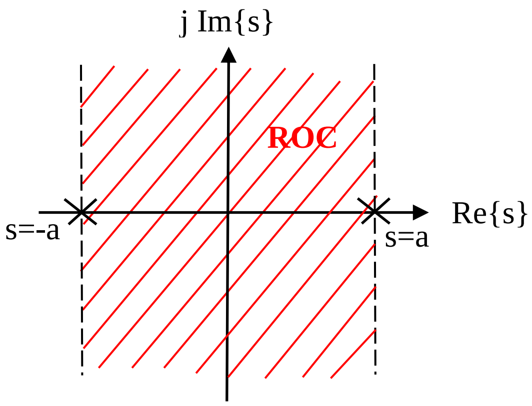
$$= \frac{e^{0} - e^{-\infty}}{a-s} + \frac{e^{-\infty} - e^{0}}{-(a+s)}; a-s > 0 \;\&\; a+s > 0$$

$$= \frac{1-0}{a-s} + \frac{0-1}{-(a+s)};\; a > s, s < a \;\&\; s > -a$$

$$= \frac{1}{a-s} + \frac{1}{a+s} = \frac{2a}{a^2 - s^2};\; -a < s < a$$

$LT[e^{-a\|t\|}] = X(s) = \dfrac{2a}{a^2 - s^2}$	ROC
	$-a < s < a$

Pole-Zero Plot with ROC:



Note: X(s) has two pole, which are located at s = a and s = -a.

5. Region of Convergence (ROC) in s-domain and Properties:

The range of values of s for which the basic definition of Laplace transform will converges or produces a finite result is called Region of Convergence (ROC).

Property-1:

If x(t) is right-sided signal with infinite duration, then its ROC is right half of the right most pole.

Ex:

$LT[e^{-at}u(t)] = X(s) = \frac{1}{s+a}$	ROC
	$s > -a$

Property-2:

If x(t) is left-sided signal with infinite duration, then its ROC is left half of the left most pole.

Ex:

$LT[e^{at}u(-t)] = X(s) = \frac{-1}{s-a}$	ROC
	$s < a$

Property-3:

If x(t) is both-sided signal with infinite duration, then its ROC is a strip, which lies between two poles.

Ex:

$LT[e^{-a\|t\|}] = X(s) = \frac{2a}{a^2 - s^2}$	ROC
	$-a < s < a$

Property-4:

If x(t) is finite duration signal, then its ROC is entire s-plane except possibly s=±∞.

Ex:

$LT[\delta(t)] = X(s) = 1$	ROC
	$Entire\ s - plaane$

Property-5:

Within the ROC, poles do not exist and ROC is independent of zero's.

Ex: Above all Examples

Property-6:

ROC is a strip, which is parallel to the jw-axes in s-plane.

Ex: Above all Examples

6. Properties of Laplace Transform:

6.1. Linear Property:

If $x_1(t)$, $x_2(t)$ are two continuous time signals and LT[$x_1(t)$] = $X_1(s)$, LT[$x_2(t)$] = $X_2(s)$,

then LT[a $x_1(t)$ + b $x_2(t)$] = a $X_1(s)$ + b $X_2(s)$ is called linear property of Laplace Transform.

Proof: From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Replace x(t) with **a $x_1(t)$ + b $x_2(t)$**

$$LT[ax_1(t) + bx_2(t)] = \int_{-\infty}^{\infty} \big(ax_1(t) + bx_2(t)\big)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} (ax_1(t)e^{-st} + bx_2(t)e^{-st})dt$$

$$= \int_{-\infty}^{\infty} ax_1(t)e^{-st}dt + \int_{-\infty}^{\infty} bx_2(t)e^{-st}dt$$

$$= a\int_{-\infty}^{\infty} x_1(t)e^{-st}dt + b\int_{-\infty}^{\infty} x_2(t)e^{-st}dt$$

$$= aLT[x_1(t)] + bLT[x_2(t)] = a\,X_1(s) + b\,X_2(s)$$

4.2. Time Shifting Property:

If x(t) is a continuous time signal and LT[x(t)] = X(s),

then LT[$x(t - t_0)$] = e^{-st_0} X(s) is called time shifting property of Laplace Transform.

Proof: From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Replace x(t) with $x(t - t_0)$

$$LT[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0)e^{-st}dt, Let\ t - t_0 = \tau, dt = d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-s(t_0+\tau)}d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-jws}e^{-s\tau}d\tau$$

$$= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}d\tau$$

$$= e^{-st_0}\,LT[x(t)]$$

$$= e^{-st_0}\,X(s)$$

4.3. Time Reversal Property:

If x(t) is a continuous time signal and LT[x(t)] = X(s),

then LT[x(-t)] = X(-s) is called time reversal property of Laplace Transform.

Proof:

From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Replace x(t) with x(-t)

$$LT[x(-t)] = \int_{-\infty}^{\infty} x(-t)e^{-st}dt\,, Let - t = \tau \Rightarrow dt = -d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-s(-\tau)}d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-(-s)\tau}d\tau\,; LT[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= X(-s)$$

4.4. Conjugate Property:

If x(t) is a continuous time signal and LT[x(t)] = X(s),

then LT[x*(t)] = X*(s*) is called conjugate property of Laplace Transform.

Proof:

From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Replace x(t) with x*(t)

$$LT[x^*(t)] = \int_{-\infty}^{\infty} x^*(t)e^{-st}dt$$

$$= \left(\int_{-\infty}^{\infty} x(t)e^{-s^*t}dt\right)^*$$

$$= (X(s^*))^*$$

$$= X^*(s^*)$$

4.5. Time Scaling Property:

If x(t) is a continuous time signal and LT[x(t)] = X(s),

then LT[x(at)] $= \frac{1}{|a|} X\left(\frac{s}{a}\right)$ is called time scaling property of Laplace Transform.

Proof:

From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Case-1 (a > 0) : Replace x(t) with x(at)

$$LT[x(at)] = \int_{-\infty}^{\infty} x(at)e^{-st}dt\,, Let\ at = \tau \Rightarrow adt = d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-s\tau/a}(d\tau/a)$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-\left(\frac{s}{a}\right)\tau}d\tau;\ LT[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= \frac{1}{a}X\left(\frac{s}{a}\right) ------(1)$$

Case-2 (a > 0): Replace x(t) with x(-at)

$$LT[x(-at)] = \int_{-\infty}^{\infty} x(-at)e^{-st}dt\,, Let - at = \tau \Rightarrow adt = -d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-s\left(-\frac{\tau}{a}\right)}(d\tau/a)$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-\left(\frac{s}{-a}\right)\tau}d\tau$$

$$= \frac{1}{a}X\left(\frac{s}{-a}\right) ----- (2)$$

Compare equations (1) and (2)

$$\Rightarrow LT[x(at)] = \frac{1}{|a|}X\left(\frac{s}{a}\right)$$

Note: If the time domain signal x(t) is scaled with ‘a’ then the frequency domain / s-domain X(s) is scaled with ‘1/a’.

4.6. Shifting in s-domain Property:

If x(t) is a continuous time signal and LT[x(t)] = X(s),

then LT[e ^{sot} x(t)] = X(s-so) is called shifting in s-domain property of Laplace Transform.

Proof: From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Replace x(t) with e ^{sot} x(t)

$$LT[e^{s_0 t}x(t)] = \int_{-\infty}^{\infty} e^{s_0 t}x(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(s-s_0)t}dt\,;\; LT[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= X(s - s_0)$$

4.7. Time Differentiation Property:

If x(t) is a continuous time causal signal and LT[x(t)] = X(s),

then LT[$\frac{d}{\mathrm{dt}}x(t)$] $= sX(s) - x(0)$ is called time differentiation property of Laplace Transform.

Proof: From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt\,, given\ that\ x(t) is\ causal$$

$$\Rightarrow LT[x(t)] = \int_{0}^{\infty} x(t)e^{-st}dt$$

Replace x(t) with $\frac{d}{\mathrm{dt}}x(t)$

$$LT\left[\frac{d}{\mathrm{dt}}x(t)\right] = \int_{0}^{\infty} \left(\frac{d}{\mathrm{dt}}x(t)\right)e^{-st}dt$$

$$= \int_{0}^{\infty} e^{-st}\left(\frac{d}{\mathrm{dt}}x(t)\right)dt$$

$$= e^{-st}x(t)\Big|_{0}^{\infty} - \int_{0}^{\infty} e^{-st}(-s)x(t)dt$$

$$= e^{-\infty}x(\infty) - e^{-0}x(0) + s\int_{0}^{\infty} x(t)e^{-st}dt$$

$$= 0 - x(0) + sX(s)$$

$$= sX(s) - x(0)$$

Note: LT $\left[\frac{d^2}{\mathrm{dt}^2}x(t)\right] = s^2X(s) - sx(0) - x'(0);\; x'(0) = \frac{d}{\mathrm{dt}}x(t), at\ t = 0$

4.8. Time Integration Property:

If x(t) is a continuous time signal and LT[x(t)] = X(s),

then LT[$\int_{-\infty}^{t} x(\tau)d\tau$] $= \frac{X(s)}{s}$ is called time integration property of Laplace Transform.

Proof:

From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Replace x(t) with $\int_{-\infty}^{t} x(\tau)d\tau$

$$LT\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} x(\tau)d\tau\right)e^{-st}dt$$

$$= \int_{-\infty}^{t} x(\tau)d\tau \frac{e^{-st}}{-s}\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)\frac{e^{-st}}{-s}dt$$

$$= 0 - 0 + \frac{1}{s}\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= \frac{X(s)}{s}$$

4.9. Differentiation in s-domain Property:

If x(t) is a continuous time signal and LT[x(t)] = X(s),

then LT[t x(t)] = - $\frac{d}{ds}X(s)$ is called differentiation in s-domain property of Laplace Transform.

Proof:

From the definition of Laplace Transform

$$LT[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Differentiate X(s) w.r.t 's'

$$\frac{d}{ds}X(s) = \int_{-\infty}^{\infty} x(t)\frac{d}{ds}e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-st}(-t)dt$$

$$= -\int_{-\infty}^{\infty} tx(t)e^{-st}dt$$

$$= -LT[tx(t)]$$

$$\Rightarrow LT[tx(t)] = -\frac{d}{ds}X(s)$$

4.10. Integration in s-domain Property:

If x(t) is a continuous time signal and LT[x(t)] = X(s),

then LT $\left[\frac{x(t)}{t}\right] = \int_s^{\infty} X(s)\text{ds}$ is called integration in s-domain property of Laplace Transform.

Proof:

From the definition of Laplace Transform

$$LT[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Integrate X(s) w.r.t 's' over the range s to ∞

$$\int_s^{\infty} X(s)\text{ds} = \int_{-\infty}^{\infty} x(t)\left(\int_s^{\infty} e^{-st}ds\right)dt$$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{e^{-st}}{-t}\Bigg|_s^{\infty}\right)dt$$

$$= \int_{-\infty}^{\infty} x(t)\frac{e^{-\infty} - e^{-0}}{-t}dt$$

$$= \int_{-\infty}^{\infty} x(t)\frac{0-1}{-t}dt$$

$$= LT\left[\frac{x(t)}{t}\right]$$

$$\Rightarrow LT\left[\frac{x(t)}{t}\right] = \int_s^{\infty} X(s)\text{ds}$$

4.11. Initial Value Theorem:

If x(t) is a continuous time causal signal and LT[x(t)] = X(s), then the initial value of a causal signal can be computed from x(t) as well as X(s) by using the formula

$$x(0) = \underset{t \to 0}{\text{Lt}} x(t) = \underset{s \to \infty}{\text{Lt}} \text{sX}(s)$$ is called initial value theorem.

Proof:

From differentiation property of Laplace Transform

$$LT\left[\frac{d}{\text{dt}}x(t)\right] = sX(s) - x(0)$$

$$\Rightarrow sX(s) = x(0) + LT\left[\frac{d}{\text{dt}}x(t)\right]$$

$$= x(0) + \int_0^{\infty}\left(\frac{d}{\text{dt}}x(t)\right)e^{-st}dt$$

Apply as limit $s \to \infty$

$$\underset{s \to \infty}{\text{Lt}} sX(s) = x(0) + \int_0^\infty \left(\frac{d}{dt}x(t)\right) e^{-\infty} dt$$

$$= x(0) + 0$$

$$\Rightarrow x(0) = \underset{t \to 0}{\text{Lt}} x(t) = \underset{s \to \infty}{\text{Lt}} sX(s)$$

4.12. Final Value Theorem:

If x(t) is a continuous time causal signal and LT[x(t)] = X(s), then the final value of a causal signal can be computed from x(t) as well as X(s) by using the formula

$$x(\infty) = \underset{t \to \infty}{\text{Lt}} x(t) = \underset{s \to 0}{\text{Lt}} sX(s)$$ is called final value theorem.

Proof:

From differentiation property of Laplace Transform

$$LT\left[\frac{d}{dt}x(t)\right] = sX(s) - x(0)$$

$$\Rightarrow sX(s) = x(0) + LT\left[\frac{d}{dt}x(t)\right]$$

$$= x(0) + \int_0^\infty \left(\frac{d}{dt}x(t)\right) e^{-st} dt$$

Apply as limit $s \to 0$

$$\underset{s \to 0}{\text{Lt}} sX(s) = x(0) + \int_0^\infty \left(\frac{d}{dt}x(t)\right) e^{-0} dt$$

$$= x(0) + \int_0^\infty \left(\frac{d}{dt}x(t)\right) dt$$

$$= x(0) + x(t)\Big|_0^\infty$$

$$= x(0) + x(\infty) - x(0)$$

$$= x(\infty)$$

$$\Rightarrow x(\infty) = \underset{t \to \infty}{\text{Lt}} x(t) = \underset{s \to 0}{\text{Lt}} sX(s)$$

4.13. Time Convolution Theorem:

If $x_1(t)$, $x_2(t)$ are two continuous time signals and LT[$x_1(t)$] = $X_1(s)$, LT[$x_2(t)$] = $X_2(s)$, then LT[$x_1(t) * x_2(t)$] = $X_1(s)\, X_2(s)$ is called time convolution theorem.

Proof:

From the definition of Laplace Transform

$$LT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Replace x(t) with $x_1(t) * x_2(t)$

$$LT[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \big(x_1(t) * x_2(t)\big)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)d\tau\right) e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left(\int_{-\infty}^{\infty} x_2(t-\tau)e^{-st}\, dt\right) d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) LT[x_2(t-\tau)]d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)e^{-s\tau}X_2(s)d\tau$$

$$= X_2(s) \int_{-\infty}^{\infty} x_1(\tau)e^{-s\tau}d\tau$$

$$= X_2(s)X_1(s)$$

$$= X_1(s)X_2(s)$$

4.14. Frequency Convolution Theorem:

If $x_1(t)$, $x_2(t)$ are two continuous time signals and LT[$x_1(t)$] = $X_1(s)$, LT[$x_2(t)$] = $X_2(s)$, then LT[$x_1(t)\, x_2(t)$] = $X_1(s) * X_2(s)$ is called frequency convolution theorem.

7. Inverse Laplace Transform:

We know that the Laplace Transform is used to convert continuous time domain signal into frequency domain or s-domain representation. Similarly, the frequency domain or s-domain representation can be converted into continuous time domain signal by using inverse Laplace Transform. Some of the formulas used in inverse Laplace Transform are

S.No.	Laplace Transform	Inverse Laplace Transform
1.	$LT[\delta(t)] = 1$	$L^{-1}[1] = \delta(t)$
2.	$LT[u(t)] = \frac{1}{s}, Re\{s\} > 0$	$L^{-1}\left[\frac{1}{s}\right] = \begin{cases} u(t), Re\{s\} > 0 \\ -u(-t), Re\{s\} < 0 \end{cases}$
3.	$LT[-u(-t)] = \frac{1}{s}, Re\{s\} < 0$	
4.	$LT[e^{-at}u(t)] = \frac{1}{s+a}, Re\{s\} > -a$	$L^{-1}\left[\frac{1}{s+a}\right] = \begin{cases} e^{-at}u(t), Re\{s\} > -a \\ -e^{-at}u(-t), Re\{s\} < -a \end{cases}$
5.	$LT[-e^{-at}u(-t)] = \frac{1}{s+a}, Re\{s\} < -a$	
6.	$LT[t^m u(t)] = \frac{m!}{s^{m+1}}, Re\{s\} > 0$	$L^{-1}\left[\frac{1}{s^m}\right] = \frac{t^{m-1}}{(m-1)!}u(t), Re\{s\} > 0$
7.	$LT[te^{-at}u(t)] = \frac{1}{(s+a)^2}, Re\{s\} > -a$	$L^{-1}\left[\frac{1}{(s+a)^2}\right] = te^{-at}u(t), Re\{s\} > -a$
8.	$LT\left[e^{-a\|t\|}\right] = \frac{2a}{a^2-s^2}, -a < Re\{s\} < a$	$L^{-1}\left[\frac{2a}{a^2-s^2}\right] = e^{-a\|t\|}, -a < Re\{s\} < a$
9.	$LT[Cos(at)u(t)] = \frac{s}{a^2+s^2}, Re\{s\} > 0$	$L^{-1}\left[\frac{s}{a^2+s^2}\right] = Cos(at)u(t), Re\{s\} > 0$
10.	$LT[Sin(at)u(t)] = \frac{a}{a^2+s^2}, Re\{s\} > 0$	$L^{-1}\left[\frac{a}{a^2+s^2}\right] = Sin(at)u(t), Re\{s\} > 0$
11.	$LT[x(t-t_0)] = e^{-st_0}X(s)$	$L^{-1}[e^{-st_0}X(s)] = x(t-t_0)$

8. Introduction to Z-Transform (ZT):

Z-Transform is a mathematical tool, which is used to evaluate z-domain representation of a discrete time domain sequence.

Z-Transform of a discrete time signal or sequence x(n) is represented with X(z) and it can be evaluated by using the formula

$$\mathrm{ZT}[\,x(n)\,] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} - - - - - - - - (1)$$

Above equation (1) is called bi-directional or both sided Z-Transform, because x(n) is both-sided.

If x(n) is causal or right sided, then its Z-Transform can be defined as

$$\mathrm{ZT}[\,x(n)\,] = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} - - - - - - - - -(2)$$

If x(n) is anti-causal or left sided, then its Z-Transform can be defined as

$$\mathrm{ZT}[\,x(n)\,] = X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n} - - - - - - - - -(3)$$

Above equations (2) and (3) are called uni-directional or one-sided Z-Transform.

Where, z is a complex variable, and it can be defined as

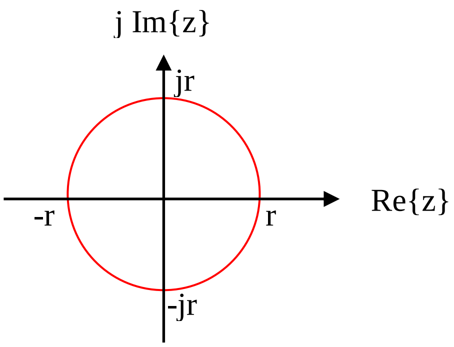
$z = r\,e^{j\omega}$

$= r\cos(\omega) + j\,r\sin(\omega)$

$= \mathrm{Re}\{\,z\,\} + j\,\mathrm{Im}\{\,z\,\}$

Where, r is magnitude of z and ω is phase of z or digital frequency, measured in rad/sample.

A graph, which is drawn between Re{ z } = r cos(ω) on x-axis and jIm{ z } = jr sin(ω) on y-axis is called z-plane.



ω	Re{z}=rcos(ω)	Im{z}=jrsin(ω)	|z|
0^0	r	j0	r
90^0	0	jr	r
180^0	– r	j0	r
270^0	0	–jr	r

z-plane is a circle, centered about origin with a radius of |z|

9. Z-Transform of various classes of Signals:

9.1. Right-sided Signal with Infinite Duration, x(n) = a ⁿ u(n)

From the basic definition of Z-Transform,

$$\mathrm{ZT}[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\mathrm{ZT}[\,a^{n}\,u(n)\,] = \sum_{n=-\infty}^{\infty} a^{n}\,u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{n}\,z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^{2} + \left(\frac{a}{z}\right)^{3} + \ldots\ldots + \left(\frac{a}{z}\right)^{\infty}$$

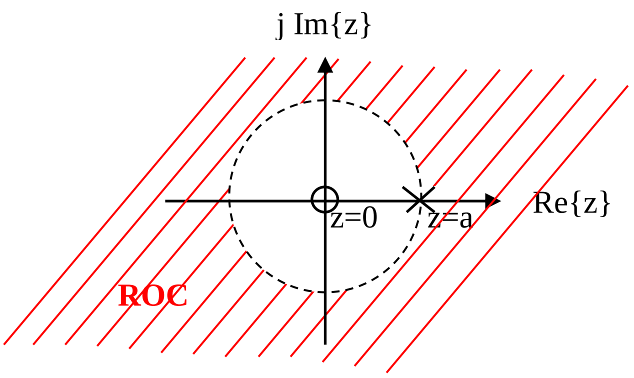
$$= \frac{1}{1-\frac{a}{z}}\ , \text{ if } \left|\frac{a}{z}\right| < 1$$

$$= \frac{1}{\frac{z-a}{z}}\ , \quad |a| < |z|$$

$$X(z) = \frac{z}{z-a}\ , \quad |z| > |a|$$

	ROC
$ZT[a^n u(n)] = X(z) = \dfrac{z}{z-a}$	$\lvert z \rvert > \lvert a \rvert$

Pole-Zero Plot with ROC:



Note: X(z) has one zero, which is located at z=0 and one pole, which is located at z=a.

9.2. Left-sided Signal with Infinite Duration, $x(n) = -a^n u(-n-1)$:

From the basic definition of z transform

$$ZT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$ZT[-a^n u(-n-1)] = \sum_{n=-\infty}^{\infty} [-a^n u(-n-1)]\, z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \left(\frac{a}{z}\right)^n$$

$$= -\sum_{n=1}^{\infty} \left(\frac{z}{a}\right)^n$$

$$= -\left[\left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \ldots\ldots + \left(\frac{z}{a}\right)^\infty\right]$$

$$= -\left(\frac{z}{a}\right)\left[1 + \left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \ldots\ldots + \left(\frac{z}{a}\right)^\infty\right]$$

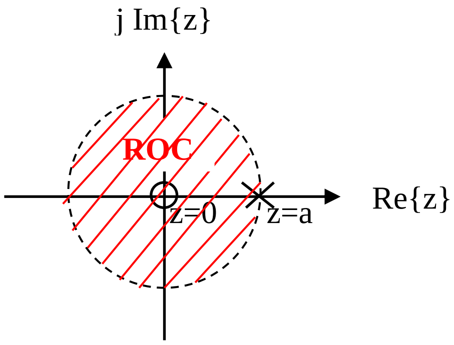
$$= -\left(\frac{z}{a}\right)\left(\frac{1}{1-\frac{z}{a}}\right), \text{ if } \left|\frac{z}{a}\right| < 1$$

$$= -\left(\frac{z}{a}\right)\left(\frac{a}{a-z}\right), \text{ if } |z| < |a|$$

$$= \frac{z}{z-a}, \quad |z| < |a|$$

$ZT[-a^n u(-n-1)] = X(z) = \dfrac{z}{z-a}$	ROC
	$\lvert z \rvert < \lvert a \rvert$

Pole-Zero Plot with ROC:



Note: X(z) has one zero, which is located at z=0 and one pole, which is located at z=a.

9.3. Both-sided Signal with Infinite Duration, x(n) = a $^{|n|}$:

From the basic definition of z transform

$$ZT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$ZT[\,a^{|n|}\,] = \sum_{n=-\infty}^{\infty} a^{|n|}\; z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{|n|}\; z^{-n} + \sum_{n=0}^{\infty} a^{|n|}\; z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n}\; z^{-n} + \sum_{n=0}^{\infty} a^{n}\; z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (az)^{-n} + \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}$$

$$= \sum_{n=1}^{\infty} (az)^{n} + \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}$$

$$= \left[az + (az)^2 + (az)^3 + \ldots\ldots + (az)^{\infty}\right] + \left[1 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \ldots\ldots + \left(\frac{a}{z}\right)^{\infty}\right]$$

$$= az\left[1 + az + (az)^2 + (az)^3 + \ldots\ldots + (az)^{\infty}\right] + \left[1 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \ldots\ldots + \left(\frac{a}{z}\right)^{\infty}\right]$$

$$= az\left[\frac{1}{1-az}\right] + \left[\frac{1}{1-\frac{a}{z}}\right], \quad \text{if} \quad |az| < 1 \;\; \& \;\; \left|\frac{a}{z}\right| < 1$$

$$= \left[\frac{az}{1-az}\right] + \left[\frac{z}{z-a}\right], \quad \text{if} \quad |z| < 1/|a| \;\; \& \;\; |a| < |z|$$

$$= \left[\frac{az(z-a) + z(1-az)}{(1-az)(z-a)}\right], \quad \text{if} \quad |z| < 1/|a| \;\; \& \;\; |z| > |a|$$

$$= \left[\frac{z(az - a^2 + 1 - az)}{(1-az)(z-a)}\right], \quad |a| < |z| < 1/|a|$$

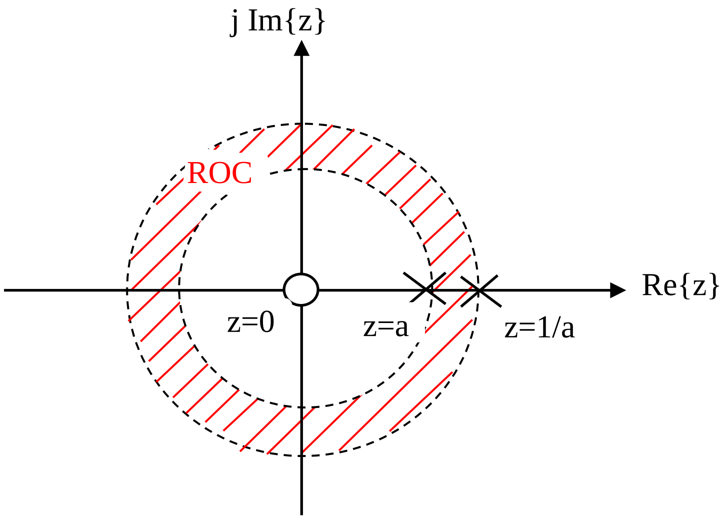
$$= \left[\frac{z(1-a^2)}{(1-az)(z-a)}\right], \quad |a| < |z| < 1/|a|$$

$$= \left[\frac{z(1-a^2)}{-a(z-1/a)(z-a)}\right], \quad |a| < |z| < 1/|a|$$

$$= \left[\frac{z(a-1/a)}{(z-1/a)(z-a)}\right], \quad |a| < |z| < 1/|a|$$

	ROC
$ZT[a^{\lvert n \rvert}] = X(z) = \dfrac{z\left(a - \frac{1}{a}\right)}{(z-a)\left(z - \frac{1}{a}\right)}$	$\lvert a \rvert < \lvert z \rvert < \dfrac{1}{\lvert a \rvert}$

Pole-Zero Plot with ROC:



Note: X(z) has one zero, which is located at z=0 and two poles, which is located at z=a and z=1/a.

9.4. Finite duration Signal:

Example-1: x(n) = {1}	**Example-2:** x(n) = {1,-1}	**Example-3:** $x(n) = \{1, -\underset{\uparrow}{1}\}$
From the definition of z-transform $ZT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ $= x(0)z^{-0}$ $= 1 \times 1$ $= 1$	From the basic definition of z-transform $ZT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ $= x(0)z^{-0} + x(1)z^{-1}$ $= 1\times 1 - 1 \times z^{-1}$ $= 1 - z^{-1}$ $= \frac{z-1}{z}$	From the basic definition of z-transform $ZT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ $= x(-1)z^{-(-1)} + x(0)z^{-0}$ $= 1 \times z - 1 \times z^{0}$ $= z - 1$

	ROC
$ZT[x(n)] = X(z) = 1$	Entire z-plane

	ROC
$ZT[x(n)] = X(z) = \dfrac{z-1}{z}$	Entire z-plane except z=0

	ROC
$ZT[x(n)] = X(z) = z - 1$	Entire z-plane except z=±∞

10. Region of Convergence (ROC) in z-domain and Properties:

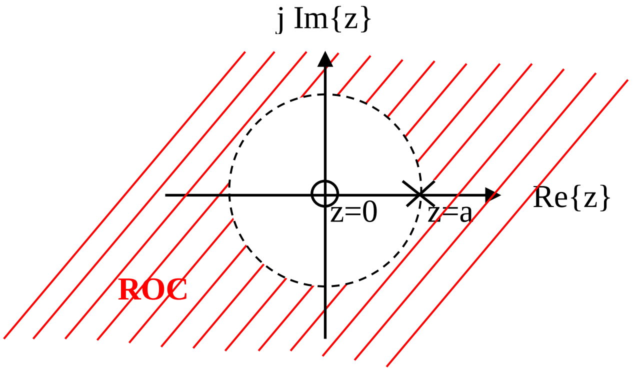
The range of values of z for which the basic definition of z transform will converges or produces a finite result is called Region of Convergence (ROC).

Property-1:

If x(n) is right-sided sequence with infinite duration, then its ROC is outside the circle of outermost pole.

Ex:

	ROC
$ZT[a^n u(n)] = X(z) = \frac{z}{z-a}$	$\lvert z \rvert > \lvert a \rvert$

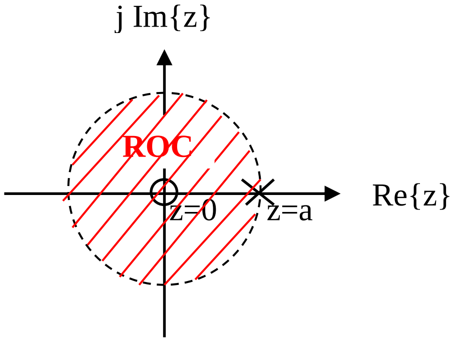


Property-2:

If x(n) is left-sided sequence with infinite duration, then its ROC is inside the circle of innermost pole.

Ex:

	ROC
$ZT[-a^n u(-n-1)] = X(z) = \frac{z}{z-a}$	$\lvert z \rvert < \lvert a \rvert$

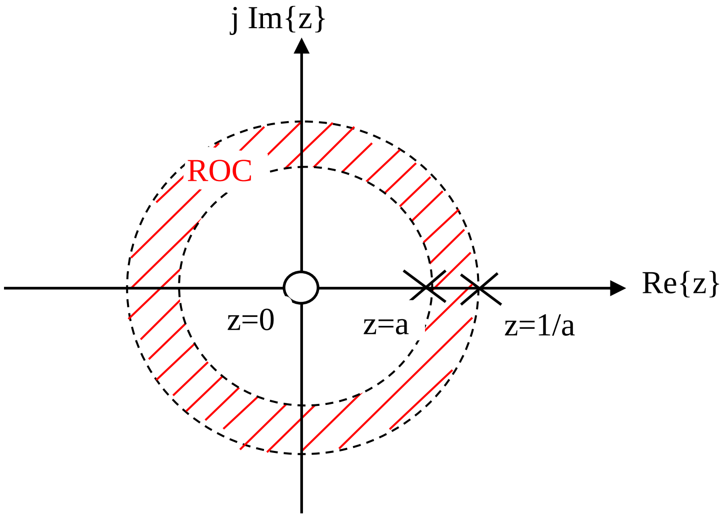


Property-3:

If x(n) is both-sided sequence with infinite duration, then its ROC is a finite duration ring, which lies between two poles.

Ex:

	ROC
$ZT[a^{\lvert n \rvert}] = X(z) = \dfrac{z\left(a-\frac{1}{a}\right)}{(z-a)\left(z-\frac{1}{a}\right)}$	$\lvert a \rvert < \lvert z \rvert < \dfrac{1}{\lvert a \rvert}$



Property-4:

If x(n) is finite duration sequence, then its ROC is entire z-plane except possibly z=0 and/or z=±∞.

Ex:

	ROC
$ZT[x(n)] = X(z) = 1$	Entire z-plane
$ZT[x(n)] = X(z) = \dfrac{z-1}{z}$	Entire z-plane except z=0
$ZT[x(n)] = X(z) = z-1$	Entire z-plane except z=±∞

Property-5:

Within the ROC, poles do not exist.

Ex: Above all examples.

Property-6:

ROC is independent of zero's.

Ex: Above all examples.

11. Properties of Z Transform:

11.1. Linear Property:

If $x_1(n)$, $x_2(n)$ are two discrete time sequences and ZT[$x_1(n)$] = $X_1(z)$, ZT[$x_2(n)$] = $X_2(z)$,

then ZT[a $x_1(n)$ + b $x_2(n)$] = a $X_1(z)$ + b $X_2(z)$ is called linear property of z transform

Proof: From the basic definition of z transform of a sequence x(n)

$$ZT[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

replace x(n) with a $x_1(n)$ + b $x_2(n)$

$$\begin{aligned}
ZT[a\, x_1(n) + b\, x_2(n)] &= \sum_{n=-\infty}^{\infty} [a\, x_1(n) + b\, x_2(n)]\, z^{-n} \\
&= \sum_{n=-\infty}^{\infty} [a\, x_1(n) + b\, x_2(n)]\, z^{-n} \\
&= \sum_{n=-\infty}^{\infty} [a\, x_1(n)\, z^{-n} + b\, x_2(n)\; z^{-n}] \\
&= \sum_{n=-\infty}^{\infty} [a\, x_1(n)\, z^{-n}] + \sum_{n=-\infty}^{\infty} [b\, x_2(n)\; z^{-n}] \\
&= a \sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n} + b \sum_{n=-\infty}^{\infty} [x_2(n)\; z^{-n}] \\
&= a\, ZT[\; x_1(n)] + b\, ZT[\; x_2(n)] \\
&= a\, X_1(z) \; + b\, X_2(z)
\end{aligned}$$

11.2. Time Shifting Property:

If x(n) is a discrete time sequence and ZT[x(n)] = X(z),

then ZT[x(n – n_0)] = z $^{-no}$ X(z) is called time shifting property of z transform.

Proof: From the basic definition of z transform of a sequence x(n)

$$ZT[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\begin{aligned}
ZT[\, x(n - n_0)\,] &= \sum_{n=-\infty}^{\infty} x(n - n_0) z^{-n}, \text{Let } n - n_0 = m \Rightarrow n = n_0 + m \\
&= \sum_{m=-\infty}^{\infty} x(m) z^{-(n_0+m)} \\
&= \sum_{m=-\infty}^{\infty} x(m) z^{-n_0}\; z^{-m} \\
&= z^{-n_0} \sum_{m=-\infty}^{\infty} x(m)\; z^{-m} \\
&= z^{-n_0} ZT[\, x(n)\,] \\
&= z^{-n_0} X(z)
\end{aligned}$$

11.3. Time Reversal Property:

If x(n) is a discrete time sequence and ZT[x(n)] = X(z),

then ZT[x(– n)] = X(1/z) is called time reversal property of z transform.

Proof: From the basic definition of z transform of a sequence x(n)

$$ZT[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\begin{aligned} ZT[\, x(-n)\,] &= \sum_{n=-\infty}^{\infty} x(-n) z^{-n}, \text{Let } n = -m, n = -m, \\ &= \sum_{m=\infty}^{-\infty} x(m) z^{-(-m)} \\ &= \sum_{m=\infty}^{-\infty} x(m)\ (z^{-1})^{-m} \\ &= \sum_{m=\infty}^{-\infty} x(m) \left(\frac{1}{z}\right)^{-m} \\ &= ZT[\, x(n)\,] \text{ with replacement of } z = 1/z \\ &= X\left(\frac{1}{z}\right) \end{aligned}$$

11.4. Conjugate Property:

If x(n) is discrete time sequence and ZT[x(n)] = X(z),

then ZT[x*(n)] = X*(z*) is conjugate property of z transform.

Proof: From the basic definition of z transform of a sequence x(n)

$$ZT[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\begin{aligned} ZT[\, x^*(n)\,] &= \sum_{n=-\infty}^{\infty} x^*(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x^*(n) \left((z^*)^{-n}\right)^* \\ &= \sum_{n=-\infty}^{\infty} [\, x(n)\,(z^*)^{-n}\,]^* \\ &= \left[\sum_{n=-\infty}^{\infty} x(n)\,(z^*)^{-n}\right]^* \\ &= (ZT\,[\, x(n)\,] \text{ with } z = z^*)^* \\ &= (X(z) \text{ with } z = z^*)^* \\ &= [X(z^*)\,]^* \\ &= X^*(z^*) \end{aligned}$$

11.5. Exponential or Scaling in z-domain Property:

If x(n) is a discrete time sequence and ZT[x(n)] = X(z),

then ZT[a^n x(n)] = X(z/a) is called exponential or scaling in z-domain property.

Proof: From the basic definition of z transform of a sequence x(n)

$$\text{ZT}[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Replace x(n) with a^n x(n)

$$\text{ZT}[\, a^n\, x(n)\,] = \sum_{n=-\infty}^{\infty} a^n\, x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, a^n\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right)$$

11.6. Multiplication by n or Differentiation in z-domain Property:

If x(n) is a discrete time sequence and ZT[x(n)] = X(z),

then ZT[nx(n)]= - d/dz [X(z)] is called multiplication by n or differentiation in z domain property.

Proof:

From the basic definition of z transform of a sequence x(n)

$$\text{ZT}[\, x(n)\,] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Differentiate w.r.t z

$$\frac{d}{dz}[\, X(z)\,] = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-n) z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-n) z^{-n} z^{-1}$$

$$= -z^{-1}\sum_{n=-\infty}^{\infty} [\, n\, x(n)\,]\, z^{-n}$$

$$\frac{d}{dz}[\, X(z)\,] = -\frac{1}{z}\,\text{ZT}[\, nx(n)\,]$$

$$\text{ZT}[\, nx(n)\,] = -z\frac{d}{dz}[\, X(z)\,]$$

11.7. Initial Value Theorem:

If x(n) is a discrete time causal sequence and ZT[x(n)] = X(z), then the initial value of a causal signal can be computed from x(n) as well as X(z) by using the formula

$$x(0) = \underset{n \to 0}{\text{Lt}}\, x(n) = \underset{z \to \infty}{\text{Lt}}\, X(z) \text{ is called initial value theorem.}$$

Proof:

From the basic definition of z transform of a sequence x(n)

$$\text{ZT}[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots\dots\dots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots\dots\dots$$

Apply as limit $z \to \infty$

$$\underset{z \to \infty}{\text{Lt}}\, X(z) = x(0) + \frac{x(1)}{\infty} + \frac{x(2)}{\infty^2} + \dots\dots\dots$$

$$= x(0) + 0 + 0 + \dots\dots\dots\dots$$

$$= x(0)$$

$$\Rightarrow x(0) = \underset{n \to 0}{\text{Lt}}\, x(n) = \underset{z \to \infty}{\text{Lt}}\, X(z)$$

11.8. Final Value Theorem:

If x(n) is a discrete time causal sequence and ZT[x(n)] = X(z), then the final value of a causal signal can be computed from x(n) as well as X(z) by using the formula

$$x(\infty) = \underset{n \to \infty}{\text{Lt}}\, x(n) = \underset{z \to 1}{\text{Lt}}\, (1-z^{-1})X(z) = \underset{z \to 1}{\text{Lt}}\, (z-1)X(z) \text{ is called final value theorem.}$$

Proof: From the basic definition of z transform of a causal sequence x(n)

$$\text{ZT}[\,x(n)\,] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Replace x(n) by x(n) – x(n – 1)

$$\text{ZT}[\,x(n) - x(n-1)] = \sum_{n=0}^{\infty} [\,x(n) - x(n-1)\,] z^{-n}$$

$$X(z) - z^{-1}X(z) = \sum_{n=0}^{\infty} [\,x(n) - x(n-1)\,] z^{-n}$$

$$(1 - z^{-1})X(z) = \sum_{n=0}^{\infty} [\,x(n) - x(n-1)\,] z^{-n}$$

Apply as $z \to 1$

$$\underset{z \to 1}{Lt}(1 - z^{-1})X(z) = \underset{z \to 1}{Lt}\sum_{n=0}^{\infty}[\,x(n) - x(n-1)\,]z^{-n}$$

$$= \sum_{n=0}^{\infty}[\,x(n) - x(n-1)\,]\underset{z \to 1}{Lt}z^{-n}$$

$$= \sum_{n=0}^{\infty}[\,x(n) - x(n-1)\,]$$

$$= [x(0) - x(-1)] + [x(1) - x(0)] + [x(2) - x(1)] + \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + [x(\infty - 1) - x(\infty - 2)] + [x(\infty) - x(\infty - 1)]$$

$$= -x(-1) + x(\infty)$$

$$= x(\infty)$$

$$\Rightarrow x(\infty) = \underset{n \to \infty}{Lt}x(n) = \underset{z \to 1}{Lt}(1 - z^{-1})X(z) = \underset{z \to 1}{Lt}(z - 1)X(z)$$

11.9. Time Convolution Theorem:

If $x_1(n)$, $x_2(n)$ are two discrete time sequences and ZT[$x_1(n)$] = $X_1(z)$, ZT[$x_2(n)$] = $X_2(z)$, then ZT[$x_1(n) * x_2(n)$] = $X_1(z)\,X_2(z)$ is called time convolution theorem.

Proof:

From the basic definition of z transform of a sequence x(n)

$$ZT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Replace x(n) by $x_1(n) * x_2(n)$

$$ZT[\,x_1(n) * x_2(n)\,] = \sum_{n=-\infty}^{\infty}[\,x_1(n) * x_2(n)\,]\,z^{-n}$$

$$= \sum_{n=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty}[\,x_1(m)\,x_2(n-m)\,]\right)z^{-n}$$

change the order of summation

$$= \sum_{m=-\infty}^{\infty} x_1(m)\left(\sum_{n=-\infty}^{\infty} x_2(n-m)\,z^{-n}\right)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\,(ZT[x_2(n-m)])$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\,z^{-m}ZT[x_2(n)]$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\,z^{-m}X_2(z)$$

$$= X_1(z)X_2(z)$$

12. Inverse Z-Transform:

Inverse z-transform is used to evaluate the discrete time sequence x(n) from the z-domain X(z) and its Region of Convergence (ROC). Various methods of Inverse z transform are given below.

- Partial Fractions Method
- Power Series Method or Long Division Method
- Residue Method or Contour Integral Method

12.1. Partial Fractions Method:

In this method, take X(z)/z and split into partial fractions and finally multiply with z and use the following formulas to obtain the time domain sequence x(n).

- $ZT[\delta(n)] = 1 \Rightarrow Z^{-1}[1] = \delta(n)$
- $ZT[\delta(n-m)] = z^{-m} \Rightarrow Z^{-1}[z^{-m}] = \delta(n-m)$
- $ZT[x(n-m)] = z^{-m} X(z) \Rightarrow Z^{-1}[z^{-m} X(z)] = x(n-m)$
- $\begin{matrix} ZT[a^n u(n)] = \frac{z}{z-a}, \; |z| > a \\ ZT[-a^n u(-n-1)] = \frac{z}{z-a}, \; |z| < a \end{matrix} \Rightarrow Z^{-1}\left[\frac{z}{z-a}\right] = \begin{cases} a^n u(n)], & if \; |z| > a \\ -a^n u(-n-1)], & if \; |z| < a \end{cases}$
- $ZT[n\,a^n u(n)] = \frac{az}{(z-a)^2}, |z| > a \Rightarrow Z^{-1}\left(\frac{az}{(z-a)^2}\right) = n\,a^n u(n), if \; |z| > a$
- $ZT[n(n-1)\,a^n u(n)] = \frac{a^2 z}{(z-a)^3}, |z| > a \Rightarrow Z^{-1}\left(\frac{a^2 z}{(z-a)^3}\right) = n(n-1)\,a^n u(n), if \; |z| > a$

Example:

$$X(z) = \frac{N(z)}{(z-p_1)(z-p_2)(z-p_3)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{N(z)}{z(z-p_1)(z-p_2)(z-p_3)}$$

$$\frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z-p_1} + \frac{C}{z-p_2} + \frac{D}{z-p_3}$$

$$X(z) = A + B\left(\frac{z}{z-p_1}\right) + C\left(\frac{z}{z-p_2}\right) + D\left(\frac{z}{z-p_3}\right)$$

$$x(n) = Z^{-1}\left[A + B\left(\frac{z}{z-p_1}\right) + C\left(\frac{z}{z-p_2}\right) + D\left(\frac{z}{z-p_3}\right)\right]$$

$$x(n) = A\,Z^{-1}(1) + B\,Z^{-1}\left(\frac{z}{z-p_1}\right) + C\,Z^{-1}\left(\frac{z}{z-p_2}\right) + D\,Z^{-1}\left(\frac{z}{z-p_3}\right)$$

$$= A\delta(n) + Bp_1^n u(n) + Cp_2^n u(n) + Dp_3^n u(n)$$

12.2. Power Series or Long Division Method:

Partial fraction method is not suitable to evaluate the time domain sequence x(n) when the z-domain X(z) consists of one pole or the factorization of denominator part of X(z) is not possible, to solve such problems, the power series or long division method is used. Process of power series or long division method is given below.

Case 1:

To obtain the causal or right sided sequence, assume x(n) = 0, n < 0.

From the basic definition of z transform

$$ZT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + ...(1)$$

It is the negative power series expansion of X(z)

From given X(z) = N(z) / D(z), determine the negative power series polynomial by using long division method.

$$X(z) = N(z) / D(z) = a + b\,z^{-1} + c\,z^{-2} + ...\ (2)$$

Now compare equations 1 & 2, implies

$$x(n) = \{\underset{\uparrow}{a}, b, c, d, ...\}$$

It is the sequence representation of required discrete time domain signal

Case 2:

To obtain the anti-causal or left sided sequence, assume x(n) = 0, n > 0.

From the basic definition of z transform

$$ZT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n}$$

$$X(z) = x(-1)\,z + x(-2)\,z^{2} + x(-3)\,z^{3} + ...(1)$$

It is the positive power series expansion of X(z)

From given X(z) = N(z) / D(z), determine the positive power series polynomial by using long division method.

$$X(z) = N(z) / D(z) = a\,z + b\,z^{2} + c\,z^{3} + ...\ (2)$$

Now compare equations 1 & 2, implies $x(n) = \{....................d, c, b, a, \underset{\uparrow}{0}\}$

12.3. Residue Method or Contour Integral Method:

If the z-domain X(z) has multiple poles at a single location, then residue or contour integral method is convenient to evaluate discrete time sequence x(n).

$$If\ X(z) = \frac{p(z)}{(z-a)^N}, \qquad then\ x(n) = \frac{1}{(N-1)!} \ \underset{z \to a}{Lt} \left[\frac{d^{N-1}}{dz^{N-1}} \left(p(z)\ z^{n-1} \right) \right]$$

Where,

p(z) : Numerator polynomial of X(z)

z = a : Location of pole

N : Number of poles located at z = a.

13. Solved Problems:

13.1. Determine the Laplace Transform of $x(t) = e^{-t}u(t) + e^{-3t}u(t)$

$$LT[x(t)] = LT[e^{-t}u(t) + e^{-3t}u(t)]$$

$$= \frac{1}{s+1} + \frac{1}{s+3}, Re\{s\} > -1 \ \& \ Re\{s\} > -3$$

$$= \frac{2(s+2)}{(s+1)(s+3)}, Re\{s\} > -1$$

13.2. Determine the Laplace Transform of $x(t) = e^{-t}u(-t) + e^{-3t}u(-t)$

$$LT[x(t)] = LT[e^{-t}u(-t) + e^{-3t}u(-t)]$$

$$= \frac{-1}{s+1} + \frac{-1}{s+3}, Re\{s\} < -1 \ \& \ Re\{s\} < -3$$

$$= \frac{-2(s+2)}{(s+1)(s+3)}, Re\{s\} < -3$$

13.3. Determine the Laplace Transform of $x(t) = e^{-t}u(-t) + e^{-3t}u(t)$

$$LT[x(t)] = LT[e^{-t}u(-t) + e^{-3t}u(t)]$$

$$= \frac{-1}{s+1} + \frac{1}{s+3}, Re\{s\} < -1 \ \& \ Re\{s\} > -3$$

$$= \frac{-2}{(s+1)(s+3)}, -3 < Re\{s\} < -1$$

13.4. Determine the Laplace Transform of $x(t) = e^{-t}u(t) + e^{-3t}u(-t)$

$$LT[x(t)] = LT[e^{-t}u(t) + e^{-3t}u(-t)]$$

$$= \frac{1}{s+1} + \frac{-1}{s+3}, Re\{s\} > -1 \ \& \ Re\{s\} < -3$$

$$= \frac{2}{(s+1)(s+3)}, Does\ Not\ Exist$$

13.5. Determine the Laplace Transform of $x(t) = 3e^{-2|t|} - 2e^{-3|t|}$

$$LT[x(t)] = LT\left[3e^{-2|t|} - 2e^{-3|t|}\right]$$

$$= 3\frac{2(2)}{2^2 - s^2} - 2\frac{2(3)}{3^2 - s^2}, -2 < Re\{s\} < 2 \,\&\, -3 < Re\{s\} < 3$$

$$= \frac{12}{4 - s^2} - \frac{12}{9 - s^2}$$

$$= 12\frac{9 - s^2 - (4 - s^2)}{(4 - s^2)(9 - s^2)}$$

$$= \frac{60}{(4 - s^2)(9 - s^2)}, -2 < Re\{s\} < 2$$

13.6. Determine the Laplace Transform of $x(t) = e^{-2t}u(t-3)$

$$LT[x(t)] = LT[e^{-2t}u(t-3)]$$

$$= LT\left[e^{-2(t-3+3)}u(t-3)\right]$$

$$= e^{-6}LT[e^{-2(t-3)}u(t-3)]$$

$$= e^{-6}e^{-3s}\frac{1}{s+2}$$

$$= \frac{e^{-3(s+2)}}{s+2}, Re\{s\} > -2$$

13.7. Determine the Laplace Transform of $x(t) = 2Cos\left(6t + \frac{\pi}{4}\right)u(t)$

$$LT[x(t)] = LT\left[2Cos\left(6t + \frac{\pi}{4}\right)u(t)\right]$$

$$= LT\left[2\left(Cos(6t)Cos\left(\frac{\pi}{4}\right) - Sin(6t)Sin\left(\frac{\pi}{4}\right)\right)u(t)\right]$$

$$= LT\left[\sqrt{2}(Cos(6t) - Sin(6t))u(t)\right]$$

$$= \sqrt{2}\left(\frac{s}{6^2 + s^2} - \frac{6}{6^2 + s^2}\right)$$

$$= \frac{\sqrt{2}(s-6)}{36 + s^2}, Re\{s\} > 0$$

13.8. Determine the Laplace Transform of $x(t) = 4Cos^2\left(3t + \frac{\pi}{8}\right)u(t)$

$$LT[x(t)] = LT\left[4Cos^2\left(3t + \frac{\pi}{8}\right)u(t)\right]$$

$$= LT\left[2 + 2Cos\left(6t + \frac{\pi}{4}\right)u(t)\right]$$

$$= LT[2 + \sqrt{2}(Cos(6t) - Sin(6t))u(t)]$$

$$= \frac{2}{s} + \sqrt{2}\left(\frac{s}{6^2 + s^2} - \frac{6}{6^2 + s^2}\right)$$

$$= \frac{2}{s} + \frac{\sqrt{2}(s-6)}{36 + s^2}, Re\{s\} > 0$$

13.9. Determine the Laplace Transform of $y(t) = te^{-2|t|}$

$$LT[y(t)] = LT[tx(t)], \boldsymbol{x}(t) = e^{-2|t|}$$

$$= -\frac{d}{ds}X(s)$$

$$= -\frac{d}{ds}\left(\frac{2(2)}{2^2 - s^2}\right)$$

$$= -\frac{d}{ds}\left(\frac{4}{4 - s^2}\right)$$

$$= -\left(\frac{-4(-2s)}{(4 - s^2)^2}\right)$$

$$= \frac{-8s}{(4 - s^2)^2}, -2 < Re\{s\} < 2$$

13.10. Determine the Laplace Transform of $y(t) = tSin(2t)u(t)$

$$LT[y(t)] = LT[tx(t)]$$

$$= -\frac{d}{ds}X(s)$$

$$= -\frac{d}{ds}\left(\frac{2}{2^2 + s^2}\right)$$

$$= -\frac{d}{ds}\left(\frac{2}{4 + s^2}\right)$$

$$= -\left(\frac{-2(2s)}{(4 + s^2)^2}\right)$$

$$= \frac{4s}{(4 + s^2)^2}, Re\{s\} > 0$$

13.11. Determine the Laplace Transform of $y(t) = tCos(2t)u(t)$

$$LT[y(t)] = LT[tx(t)]$$

$$= -\frac{d}{ds}X(s)$$

$$= -\frac{d}{ds}\left(\frac{s}{2^2 + s^2}\right)$$

$$= -\frac{d}{ds}\left(\frac{s}{4 + s^2}\right)$$

$$= -\left(\frac{(4 + s^2)1 - s(2s)}{(4 + s^2)^2}\right)$$

$$= -\left(\frac{4 + s^2 - 2s^2}{(4 + s^2)^2}\right)$$

$$= \frac{-4 + s^2}{(4 + s^2)^2}, Re\{s\} > 0$$

13.12. Determine the Laplace Transform of $y(t) = e^{-at}Sin(bt)u(t)$

$$LT[Sin(bt)u(t)] = \frac{b}{b^2 + s^2}, Re\{s\} > 0$$

$$\Rightarrow LT[y(t)] = LT[e^{-at}Sin(bt)u(t)]$$

$$= \frac{b}{b^2 + (s + a)^2}, Re\{s\} > -a$$

13.13. Determine the Laplace Transform of $y(t) = e^{-at}Cos(bt)u(t)$

$$LT[Cos(bt)u(t)] = \frac{s}{b^2 + s^2}, Re\{s\} > 0$$

$$\Rightarrow LT[y(t)] = LT[e^{-at}Cos(bt)u(t)]$$

$$= \frac{s + a}{b^2 + (s + a)^2}, Re\{s\} > -a$$

13.14. Determine the Laplace Transform of $y(t) = \frac{e^{-at}-e^{-bt}}{t}u(t)$

$$LT[y(t)] = LT\left[\frac{x(t)}{t}\right]$$

$$= \int_s^\infty X(s)ds$$

$$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= log|s+a| - log|s+b| \Big|_s^\infty, Re\{s\} > -a \ \& \ Re\{s\} > -b$$

$$= log\left|\frac{s+a}{s+b}\right|_s^\infty$$

$$= 0 - log\left|\frac{s+a}{s+b}\right|$$

$$= log\left|\frac{s+b}{s+a}\right|, Re\{s\} > -b, if\ a > b, \& \ Re\{s\} > -a, if\ b > a$$

13.15. Determine the Laplace Transform of $y(t) = \frac{Cos(at)-Cos(bt)}{t}u(t)$

$$LT[y(t)] = LT\left[\frac{x(t)}{t}\right]$$

$$= \int_s^\infty X(s)ds$$

$$= \int_s^\infty \left(\frac{s}{a^2+s^2} - \frac{s}{b^2+s^2}\right) ds$$

$$= \frac{1}{2}log|a^2+s^2| - log|b^2+s^2| \Big|_s^\infty, Re\{s\} > 0$$

$$= \frac{1}{2}log\left|\frac{a^2+s^2}{b^2+s^2}\right|_s^\infty$$

$$= 0 - \frac{1}{2}log\left|\frac{a^2+s^2}{b^2+s^2}\right|$$

$$= \frac{1}{2}log\left|\frac{b^2+s^2}{a^2+s^2}\right|, Re\{s\} > 0$$

13.16. Determine the Laplace Transform of $y(t) = \frac{Sin(at)}{t} u(t)$

$$LT[y(t)] = LT\left[\frac{x(t)}{t}\right]$$

$$= \int_s^\infty X(s)ds$$

$$= \int_s^\infty \left(\frac{a}{a^2+s^2}\right) ds$$

$$= Tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

$$= \frac{\pi}{2} - Tan^{-1}\left(\frac{s}{a}\right)$$

$$= Cot^{-1}\left(\frac{s}{a}\right), Re\{s\} > 0$$

13.17. Find initial and final values of a signal x(t), given $X(s) = \frac{2(s+1)}{s^2+4s+7}$

Initial value:

$$x(0) = \underset{t \to 0}{\text{Lt}} x(t)$$

$$= \underset{s \to \infty}{\text{Lt}} sX(s)$$

$$= \underset{s \to \infty}{\text{Lt}} s\frac{2(s+1)}{s^2+4s+7}$$

$$= \underset{s \to \infty}{\text{Lt}} \frac{2s^2+s}{s^2+4s+7}$$

$$= 2$$

Final value:

$$x(\infty) = \underset{t \to \infty}{\text{Lt}} x(t)$$

$$= \underset{s \to 0}{\text{Lt}} sX(s)$$

$$= \underset{s \to 0}{\text{Lt}} s\frac{2(s+1)}{s^2+4s+7}$$

$$= \underset{s \to 0}{\text{Lt}} \frac{2s^2+s}{s^2+4s+7}$$

$$= 0$$

13.18. Find the value of x(t) as t→∞, given $X(s) = \frac{2}{s(1+s)}$

$$x(\infty) = \underset{t \to \infty}{\text{Lt}} x(t)$$
$$= \underset{s \to 0}{\text{Lt}} sX(s)$$
$$= \underset{s \to 0}{\text{Lt}} s\frac{2}{s(s+1)}$$
$$= \underset{s \to 0}{\text{Lt}} \frac{2}{s+1}$$
$$= 2$$

13.19. Find the final value of a signal x(t), given $X(s) = \frac{a}{s^2+a^2}, Re\{s\} > 0$

Final value:

$$x(\infty) = \underset{t \to \infty}{\text{Lt}} x(t)$$
$$= \underset{t \to \infty}{\text{Lt}} \text{Sin(at)}$$
$$= \text{Sin}(\infty)$$
$$= -1 \leq x(\infty) \leq 1$$

13.20. Determine the Laplace Transform of $x(t) = e^{-t}u(t) * e^{-3t}u(t)$

$$LT[x(t)] = LT[e^{-t}u(t) * e^{-3t}u(t)]$$
$$= \frac{1}{s+1}\frac{1}{s+3}, Re\{s\} > -1 \;\&\; Re\{s\} > -3$$
$$= \frac{1}{(s+1)(s+3)}, Re\{s\} > -1$$

13.21. Determine the Laplace Transform of $x(t) = e^{-t}u(-t) * e^{-3t}u(-t)$

$$LT[x(t)] = LT[e^{-t}u(-t) * e^{-3t}u(-t)]$$
$$= \frac{-1}{s+1}\frac{-1}{s+3}, Re\{s\} < -1 \;\&\; Re\{s\} < -3$$
$$= \frac{1}{(s+1)(s+3)}, Re\{s\} < -3$$

13.22. Determine the Laplace Transform of $x(t) = e^{-t}u(-t) * e^{-3t}u(t)$

$$LT[x(t)] = LT[e^{-t}u(-t) * e^{-3t}u(t)]$$

$$= \frac{-1}{s+1}\frac{1}{s+3}, Re\{s\} < -1 \ \& \ Re\{s\} > -3$$

$$= \frac{-1}{(s+1)(s+3)}, -3 < Re\{s\} < -1$$

13.23. Determine the Laplace Transform of $x(t) = 3e^{-2|t|} * 2e^{-3|t|}$

$$LT[x(t)] = LT\left[3e^{-2|t|} * 2e^{-3|t|}\right]$$

$$= 3\frac{2(2)}{2^2 - s^2}2\frac{2(3)}{3^2 - s^2}, -2 < Re\{s\} < 2 \ \& -3 < Re\{s\} < 3$$

$$= \frac{12}{4 - s^2}\frac{12}{9 - s^2}$$

$$= \frac{144}{(4 - s^2)(9 - s^2)}, 2 < Re\{s\} < 2$$

13.24. Determine the Inverse Laplace Transform of $X(s) = \frac{s+2}{(s+1)(s+3)}$

$$X(s) = \frac{s+2}{(s+1)(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{s+3}$$

$$= \frac{1}{2}\left(\frac{1}{s+1} + \frac{1}{s+3}\right)$$

$$A = \frac{-1+2}{-1+3} = \frac{1}{2}$$

$$B = \frac{-3+2}{-3+1} = \frac{1}{2}$$

$$x(t) = \begin{cases} \frac{1}{2}(e^{-t}u(t) + e^{-3t}u(t)), & if\ Re\{s\} > -1 \\ \frac{1}{2}(-e^{-t}u(-t) - e^{-3t}u(-t)), & if\ Re\{s\} < -3 \\ \frac{1}{2}(-e^{-t}u(-t) + e^{-3t}u(t)), if -3 < Re\{s\} < -1 & \end{cases}$$

13.25. Determine the Laplace Transform of tx(t), if $X(s) = \frac{1}{s^2+s+1}$

$$LT[tx(t)] = -\frac{d}{ds}X(s)$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2 + s + 1}\right)$$

$$= -\left(\frac{-1(2s+1)}{(s^2 + s + 1)^2}\right)$$

$$= \frac{2s+1}{(s^2 + s + 1)^2}$$

13.26. Determine the Inverse Laplace Transform of $X(s) = \frac{5-s}{s^2-s-2}$ by assuming the Fourier Transform of the signal x(t) exists.

$$X(s) = \frac{5-s}{s^2-s-2}$$
$$= \frac{5-s}{(s-2)(s+1)}$$
$$= \frac{A}{s-2} + \frac{B}{s+1}$$
$$= \frac{1}{s-2} - \frac{2}{s+1}$$

$$A = \frac{5-2}{2+1} = 1$$
$$B = \frac{5+1}{-1-2} = -2$$

Required signal should be bounded, where -1 < Re{s} < 2

$$\boldsymbol{x(t) = e^{2t}u(-t) - 2e^{-t}u(t)}$$

13.27. Find the final value of a signal x(t), given $X(s) = \frac{1}{s(s-1)}$

Initial and final value theorems are used only for causal signal, where all the poles should be left sided. But in this case, one pole is located at s=1 (right hand side).

$$X(s) = \frac{1}{s(s-1)}$$
$$= \frac{A}{s} + \frac{B}{s-1}$$
$$= \frac{-1}{s} + \frac{1}{s+1}$$

Causal signal,

$$x(t) = -u(t) + e^{t}u(t)$$
$$= (e^{t} - 1)u(t)$$
$$\Rightarrow x(\infty) = e^{\infty} - 1$$
$$= \infty$$

13.28. Evaluate the Z-Transform of $x(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$

$$ZT[\,x(n)\,] = ZT\left[3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)\right]$$

$$X(z) = 3ZT\left[\left(\frac{1}{2}\right)^n u(n)\right] - 2ZT\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$= \frac{3z}{z-\frac{1}{2}} - \frac{2z}{z-\frac{1}{3}} \quad , \; |z| > \frac{1}{2} \;\&\; |z| > \frac{1}{3}$$

$$= \frac{z(3z-1-2z+1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \; |z| > \frac{1}{2}$$

$$= \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \; |z| > \frac{1}{2}$$

	ROC
$ZT\left[3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)\right] = X(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$	$\lvert z \rvert > \frac{1}{2}$

13.29. Evaluate the Z-Transform of $x(n) = 3(2)^n u(-n-1) - 2(3)^n u(-n-1)$

$$ZT[\,x(n)\,] = ZT\left[3(2)^n u(-n-1) - 2(3)^n u(-n-1)\right]$$

$$X(z) = 3ZT\left[(2)^n u(-n-1)\right] - 2ZT\left[(3)^n u(-n-1)\right]$$

$$= 3ZT\left[(2)^n u(-n-1)\right] - 2ZT\left[(3)^n u(-n-1)\right]$$

$$= \frac{-3z}{z-2} - \frac{-2z}{z-3} \quad , \; |z| < 2 \;\&\; |z| < 3$$

$$= \frac{-z(3z-9-2z+4)}{(z-2)(z-3)}, \; |z| < 2$$

$$X(z) = \frac{-z(z-5)}{(z-2)(z-3)}, \qquad |z| < 2$$

	ROC
$ZT[3(2)^n u(-n-1) - 2(3)^n u(-n-1)] = X(z) = \frac{-z(z-5)}{(z-2)(z-3)}$	$\lvert z \rvert < 2$

13.30. Evaluate the Z-Transform of $x(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)$

$$ZT[\,x(n)\,] = ZT\left[3\left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)\right]$$

$$X(z) = 3ZT\left[\left(\frac{1}{2}\right)^n u(n)\right] - 2ZT\left[(3)^n u(-n-1)\right]$$

$$= \frac{3z}{z-\frac{1}{2}} - \frac{2(-z)}{z-3} \quad , \; |z| > \frac{1}{2} \; \& \; |z| < 3$$

$$= \frac{z(3z-9+2z-1)}{\left(z-\frac{1}{2}\right)(z-3)}, \quad \frac{1}{2} < |z| < 3$$

$$= \frac{5z(z-2)}{\left(z-\frac{1}{2}\right)(z-3)}, \quad \frac{1}{2} < |z| < 3$$

	ROC
$ZT\left[3\left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)\right] = X(z) = \dfrac{5z(z-2)}{\left(z-\frac{1}{2}\right)(z-3)}$	$\frac{1}{2} < \lvert z \rvert < 3$

13.31. Evaluate the Z-Transform of 3x1(n)+2x2(n), $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$ and $x_2(n) = \left(\frac{1}{3}\right)^n u(n)$

$$ZT[\,3x_1(n) + 2x_2(n)\,] = 3ZT[\,x_1(n)] + 2ZT[x_2(n)\,]$$

$$= 3ZT\left[\left(\frac{1}{2}\right)^n u(n)\right] + 2ZT\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$= \frac{3z}{z-\frac{1}{2}} + \frac{2z}{z-\frac{1}{3}} \quad , \; |z| > \frac{1}{2} \; \& \; |z| > \frac{1}{3}$$

$$= \frac{z(3z-1+2z-1)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \quad |z| > \frac{1}{2}$$

$$= \frac{z(5z-2)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \quad |z| > \frac{1}{2}$$

13.32. Evaluate the Z-Transform of $x(n) = 3\left(\frac{1}{2}\right)^{n+1} u(n-9)$

$$ZT\left[3\left(\frac{1}{2}\right)^{n+1} u(n-9)\right] = 3ZT\left[\left(\frac{1}{2}\right)^{n-9+10} u(n-9)\right]$$

$$= 3\left(\frac{1}{2}\right)^{10} ZT\left[\left(\frac{1}{2}\right)^{n-9} u(n-9)\right]$$

$$= 3\left(\frac{1}{2}\right)^{10} z^{-9} ZT\left[\left(\frac{1}{2}\right)^{n} u(n)\right]$$

$$= 3\left(\frac{1}{2}\right)^{10} z^{-9}\left(\frac{z}{z-\frac{1}{2}}\right), \ |z| > \frac{1}{2}$$

$$= \frac{3\left(\frac{1}{2}\right)^{10}}{z^{8}\left(z-\frac{1}{2}\right)}, \ |z| > \frac{1}{2}$$

13.33. Evaluate the Z-Transform of u(n) and u(-n)

We know that,

$$ZT[a^n u(n)] = \frac{z}{z-a} \quad , \qquad |z| > a$$

Put a = 1

$$ZT[u(n)] = \frac{z}{z-1} \quad , \qquad |z| > 1$$

Apply time shifting property of z Transform

$$ZT[u(-n)] = \frac{1/z}{1/z-1} \ , \ |1/z| > 1$$

$$= \frac{1}{1-z} \ , \ |z| < 1$$

13.34. Evaluate the Z-Transform of $na^n u(n)$

We know that,

$$ZT[a^n u(n)] = \frac{z}{z-a}\ , \qquad |z| > a$$

Apply differentiation in z domain property

$$ZT[na^n u(n)] = -z\frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$= -z\left(\frac{(z-a)(1) - z(1)}{(z-a)^2}\right)$$

$$= -z\left(\frac{z-a-z}{(z-a)^2}\right)$$

$$= \frac{az}{(z-a)^2}$$

13.35. Evaluate the initial value of a causal signal x(n) from the z domain

$$X(z) = \frac{z(5z-2)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \qquad |z| > \frac{1}{2}$$

$$x(0) = \underset{z\to\infty}{\text{Lt}}\, X(z)$$

$$= \underset{z\to\infty}{\text{Lt}} \frac{z(5z-2)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \underset{z\to\infty}{\text{Lt}} \frac{5-\frac{2}{z}}{\left(1-\frac{1}{2z}\right)\left(1-\frac{1}{3z}\right)}$$

$$= \frac{5-0}{(1-0)(1-0)}$$

$$= 5$$

13.36. Evaluate the final value of a causal signal x(n) from the z domain

$$X(z) = \frac{z(5z-2)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \qquad |z| > \frac{1}{2}$$

$$x(\infty) = \underset{z\to 1}{\text{Lt}}\,(z-1)X(z) == \underset{z\to 1}{\text{Lt}} \frac{(z-1)z(5z-2)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{(1-1)(1)(5-2)}{(1-1/2)(1-1/3)} = 0$$

13.37. Evaluate the Z-Transform of $a^n u(n) * na^n u(n)$

We know that,

$$ZT[a^n u(n)] = \frac{z}{z-a}\ , \qquad |z| > a$$

and

$$ZT[na^n u(n)] = \frac{az}{(z-a)^2}, |z| > a$$

$$ZT[a^n u(n) * na^n u(n)] = ZT[a^n u(n)]ZT[na^n u(n)]$$

$$= \frac{z}{z-a}\frac{az}{(z-a)^2}$$

$$= \frac{az^2}{(z-a)^3}, |z| > a$$

13.38. Determine the right sided or causal sequence x(n) using partial fractions method

$$X(z) = \frac{z(z+1)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$$

$$\frac{X(z)}{z} = \frac{z+1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$$

$$= \frac{A}{z - \frac{1}{2}} + \frac{B}{z - \frac{1}{4}}$$

$$A = \frac{\frac{1}{2}+1}{\frac{1}{2} - \frac{1}{4}} = \frac{\frac{3}{2}}{\frac{1}{4}} = 6$$

$$B = \frac{\frac{1}{4}+1}{\frac{1}{4} - \frac{1}{2}} = \frac{\frac{5}{2}}{\frac{-1}{4}} = -10$$

$$X(z) = 6\left(\frac{z}{z - \frac{1}{2}}\right) - 10\left(\frac{z}{z - \frac{1}{4}}\right)$$

$$x(n) = 6Z^{-1}\left(\frac{z}{z - \frac{1}{2}}\right) - 10Z^{-1}\left(\frac{z}{z - \frac{1}{4}}\right)$$

$$= 6\left(\frac{1}{2}\right)^n u(n) - 10\left(\frac{1}{4}\right)^n u(n)$$

For a causal or right sided signal

$$|z| > \frac{1}{2}$$

and

$$|z| > \frac{1}{4}$$

13.39. Determine the left sided or anti-causal sequence x(n) using partial fractions method

$$X(z) = \frac{z(z+1)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$$

$$= 6\left(\frac{z}{z - \frac{1}{2}}\right) - 10\left(\frac{z}{z - \frac{1}{4}}\right)$$

$$x(n) = 6Z^{-1}\left(\frac{z}{z - \frac{1}{2}}\right) - 10Z^{-1}\left(\frac{z}{z - \frac{1}{4}}\right)$$

$$= -6\left(\frac{1}{2}\right)^n u(-n-1) + 10\left(\frac{1}{4}\right)^n u(-n-1)$$

For anti-causal or left sided signal

$|z| < \frac{1}{2}$

and

$|z| < \frac{1}{4}$

13.40. Determine both sided sequence x(n) using partial fractions method

$$X(z) = \frac{z(z+1)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$$

$$= 6\left(\frac{z}{z - \frac{1}{2}}\right) - 10\left(\frac{z}{z - \frac{1}{4}}\right)$$

$$x(n) = 6Z^{-1}\left(\frac{z}{z - \frac{1}{2}}\right) - 10Z^{-1}\left(\frac{z}{z - \frac{1}{4}}\right)$$

$$= -6\left(\frac{1}{2}\right)^n u(-n-1) - 10\left(\frac{1}{4}\right)^n u(n)$$

For both sided signal

$|z| < \frac{1}{2}$

and

$|z| > \frac{1}{4}$

13.41. Determine the right sided or causal sequence x(n) using partial fractions method

$$X(z)=\frac{z+1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$\frac{X(z)}{z}=\frac{z+1}{z\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$=\frac{A}{z}+\frac{B}{z-\frac{1}{2}}+\frac{C}{z-\frac{1}{4}}$$

$$A=\frac{0+1}{\left(0-\frac{1}{2}\right)\left(0-\frac{1}{4}\right)}=8$$

$$B=\frac{\frac{1}{2}+1}{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)}=\frac{\frac{3}{2}}{\frac{1}{8}}=12$$

$$C=\frac{\frac{1}{4}+1}{\frac{1}{4}\left(\frac{1}{4}-\frac{1}{2}\right)}=\frac{\frac{5}{4}}{-\frac{1}{16}}=-20$$

For a causal or right sided signal

$$|z|>\frac{1}{2}\ \&\ |z|>\frac{1}{4}$$

$$X(z)=8+12\left(\frac{z}{z-\frac{1}{2}}\right)-20\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$x(n)=8Z^{-1}[1]+12Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)-20Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$=8\delta(n)+12\left(\frac{1}{2}\right)^{n}u(n)-20\left(\frac{1}{4}\right)^{n}u(n)$$

13.42. Determine the right sided or causal sequence x(n) from power series method

$$X(z) = \frac{z+1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

We know that the negative power series expansion of X(z) is

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Given,

$$X(z) = \frac{z+1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} = \frac{z+1}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

Apply long division method and evaluate the negative power series expansion of X(z)

$$z^2 - \frac{3}{4}z + \frac{1}{8}\Big)\ z\ \ +1\ \Big(z^{-1} + \frac{7}{4}z^{-2} + \frac{19}{16}z^{-3} + \ldots\ldots\ldots\ldots$$

$$\underline{z - \frac{3}{4} + \frac{1}{8}z^{-1}}$$

$$\frac{7}{4} - \frac{1}{8}z^{-1}$$

$$\underline{\frac{7}{4} - \frac{21}{16}z^{-1} - \frac{7}{32}z^{-2}}$$

$$\frac{19}{16}z^{-1} + \frac{7}{32}z^{-2}$$

$$\underline{\frac{19}{16}z^{-1} - \frac{57}{64}z^{-2} + \frac{19}{128}z^{-2}}$$

$$\frac{71}{64}z^{-2} - \frac{19}{128}z^{-2}\ldots\ldots\ldots\ldots$$

$$X(z) = z^{-1} + \frac{7}{4}z^{-2} + \frac{19}{16}z^{-3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Compare equations (1) and (2)

$$\Rightarrow x(0) = 0, x(1) = 1,\ x(2) = \frac{7}{4}, x(3) = \frac{19}{16}, \ldots\ldots\ldots\ldots$$

Sequence representation of causal sequence x(n),

$$x(n) = \left\{\underset{\uparrow}{0},\ 1,\ \frac{7}{4},\ \frac{19}{16}, \ldots\ldots\ldots\right\}$$

13.43. Determine the left sided or anti-causal sequence x(n) from power series method

$$X(z) = \frac{z+1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

We know that the positive power series expansion of X(z) is

$$X(z) = x(1)z^1 + x(2)z^2 + x(2)z^3 + \ldots\ldots\ldots\ldots\ldots (1)$$

Given

$$X(z) = \frac{z+1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} = \frac{z+1}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{1+z}{\frac{1}{8} - \frac{3}{4}z + z^2}$$

Apply long division method and evaluate the positive power series expansion of X(z)

$$\frac{1}{8} - \frac{3}{4}z + z^2 \Big) \; 1 + z \; \left(8 + 56z + 272z^2 + 1184z^3 + \ldots\ldots\ldots\right.$$

$$\underline{1 - 6z + 8z^2}$$

$$7z - 8z^2$$

$$\underline{7z - 42z^2 + 56z^3}$$

$$34z^2 - 56z^3$$

$$\underline{34z^2 - 204z^3 + 272z^4}$$

$$148z^3 - 272z^4\ldots\ldots\ldots$$

$$X(z) = 8 + 56z + 272z^2 + 1184z^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Compare equations (1) and (2)

$$\Rightarrow x(0) = 8, x(-1) = 56, \; x(-2) = 272, x(-3) = 1184, \ldots\ldots\ldots\ldots$$

Sequence representation of anti-causal sequence,

$$x(n) = \left\{\ldots\ldots\ldots\ldots\ldots 1184, 272, 56, \underset{\uparrow}{8}\right\}$$

13.44. Determine the causal signal x(n) from Residue method

$$X(z) = \frac{z}{(z-2)^3}$$

Given p(z) = z, N = 3 and a = 2.

$$x(n) = \frac{1}{(N-1)!} \underset{z \to a}{\text{Lt}} \left[\frac{d^{N-1}}{dz^{N-1}} \left(p(z)\ z^{n-1} \right) \right]$$

$$= \frac{1}{2!} \underset{z \to 2}{\text{Lt}} \left[\frac{d^2}{dz^2} \left(z\ z^{n-1} \right) \right]$$

$$= \frac{1}{2} \underset{z \to 2}{\text{Lt}} \left[\frac{d^2}{dz^2} \left(z^n \right) \right]$$

$$= \frac{1}{2} \underset{z \to 2}{\text{Lt}} \left[n\,(n-1)\, z^{n-2} \right]$$

$$= \frac{1}{2}\ n\,(n-1)\, 2^{n-2}$$

$$= \frac{n\,(n-1)\, 2^n}{8} u(n)$$

13.45. Determine the causal signal x(n) from Residue method

$$X(z) = \frac{z^2}{(z-2)^3}$$

Given p(z) = z^2, N = 3 and a = 2.

$$x(n) = \frac{1}{(N-1)!} \underset{z \to a}{\text{Lt}} \left[\frac{d^{N-1}}{dz^{N-1}} \left(p(z)\ z^{n-1} \right) \right]$$

$$= \frac{1}{2!} \underset{z \to 2}{\text{Lt}} \left[\frac{d^2}{dz^2} \left(z^2\ z^{n-1} \right) \right]$$

$$= \frac{1}{2} \underset{z \to 2}{\text{Lt}} \left[\frac{d^2}{dz^2} \left(z^{n+1} \right) \right]$$

$$= \frac{1}{2} \underset{z \to 2}{\text{Lt}} \left[(n+1)\, n\, z^{n-1} \right]$$

$$= \frac{1}{2}\ n\,(n+1)\, 2^{n-1}$$

$$= \frac{n\,(n+1)\, 2^n}{4} u(n)$$

13.46. Determine the causal signal x(n) from Residue method

$$X(z) = \frac{1}{(z-2)^4}$$

Given p(z) = 1, N = 4 and a =2.

$$x(n) = \frac{1}{(N-1)!} \underset{z \to a}{\text{Lt}} \left[\frac{d^{N-1}}{dz^{N-1}} \left(p(z)\ z^{n-1} \right) \right]$$

$$= \frac{1}{3!} \underset{z \to 2}{\text{Lt}} \left[\frac{d^3}{dz^3} \left(1\ z^{n-1} \right) \right]$$

$$= \frac{1}{6} \underset{z \to 2}{\text{Lt}} \left[\frac{d^3}{dz^3} \left(z^{n-1} \right) \right]$$

$$= \frac{1}{6} \underset{z \to 2}{\text{Lt}} \left[(n-1)(n-2)(n-3)\, z^{n-4} \right]$$

$$= \frac{1}{6} (n-1)(n-2)(n-3)\, 2^{n-4}$$

$$= \frac{(n-1)(n-2)(n-3)\, 2^{n}}{96} u(n)$$

14. Assignment Questions:

1. Determine the Laplace Transform of following signals

   (i)$x(t) = Sin(at)u(t)$ (ii)$y(t) = Cos(at)u(t)$

2. Determine the Laplace Transform and associated ROC of the signal,

$$x(t) = e^{-t}u(t) + e^{-2t}u(t) + e^{-3t}u(t)$$

3. Determine all posible signals corresponding to the s-domain

$$X(s) = \frac{s + 16}{(s + 1)(s + 2)(s + 4)(s + 8)}$$

4. Evaluate the Z-Transform and indicate the ROC for the following sequences

   (i)$x(n) = a^n Sin(n\theta)u(n)$ (ii)$y(n) = a^n Cos(n\theta)u(n)$

5. Apply properties to Evaluate the Z-Transform and associated ROC for x(n)=a^{n+10} u(n-10

6. Apply partial fractions method and compute all possible cases of x(n) from X(z)

$$X(z) = \frac{z + 1}{(z - 1/2)(z - 1/4)(z - 1/8)}$$

7. Apply power series method and Evaluate both the causal and Non-causal sequences from

$$X(z) = \frac{z^2 + z + 1}{2z^3 + 3z^2 + z + 4}$$

8. Apply Residue method and Obtain the causal sequence x(n) from $X(z) = \frac{z(z-2)}{(z-1/4)^3}$

9. Compute the initial value of a causal sequence x(n) from the z-domain

$$X(z) = \frac{z(z + 1)(z + 2)}{(2z - 1/2)(4z - 1/4)(8z - 1/8)}$$

10. Calculate the final value of a causal sequence x(n) from the z-domain

$$X(z) = \frac{z(z + 1)(z + 2)}{(z - 1)\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)\left(z - \frac{1}{8}\right)}$$

11. Find the Inverse z-transform X(z)=log(1+az^{-1})

12. Determine the z-transform of a convoluted sequence, $x(n) = u(n) * nu(n) * n^2u(n)$

13. Find a causal sequence from the z-domain

    (i)$X(z) = \frac{1}{z-1}$ (ii) $X(z) = \frac{1}{z^2(z-1)}$ (iii) $X(z) = \frac{1}{z^{99}(z-99)}$

15. Quiz Questions:

Q. No.	Question Description	Answer
1.	Laplace transform of x(t) is defined by________ (1). $X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$ (2). $X(s)=\int_{-\infty}^{\infty} x(t)e^{st}dt$ (3). $X(s)=\int_{-\infty}^{\infty} x(t)e^{-jwt}dt$ (4). None of the above	1
2.	Inverse Laplace transform of X(s) is defined by________ (1). $x(t)=\frac{1}{2}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds$ (2). $x(t)=\frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds$ (3). $x(t)=\frac{1}{2\pi}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds$ (4). None of the above	2
3.	Laplace transform and Fourier transform are equal if the value of σ is________ (1). One (2). Infinity (3). Zero (4). None of the above	3
4.	The Laplace transform of x(t) is convergence if,________ (1). $\int_{-\infty}^{\infty} \left\|x(t)e^{-st}\right\|dt = \infty$ (2). $\int_{-\infty}^{\infty} \left\|x(t)\right\|dt \geq \infty$ (3). $\int_{-\infty}^{\infty} \left\|x(t)e^{-st}\right\|dt \succ \infty$ (4). $\int_{-\infty}^{\infty} \left\|x(t)e^{-st}\right\|dt \prec \infty$	4
5.	The Laplace transform of $x(t)=e^{-at}u(t)$ is______ (1). $\frac{1}{s+a}$ (2). $\frac{1}{s-a}$ (3). $\frac{1}{(s+a)^2}$ (4). None of the above	1
6.	The range of Re{s}, for which the Laplace transform converges is called________ (1). Region of Divergence (2). Region of Convergence (3). Both 1& 2 (4). None of the above	2
7.	The Region of Convergence of signal x(t) = $x(t)=e^{-at}u(t)$ is______ (1). Re{s} = 0 (2).Re{s} = a (3). Re{s} > -a (4). Re{s} < -a	3
8.	The Region of Convergence of signal x(t) = $x(t)=-e^{-at}u(-t)$ is______ (1). Re{s} = 0 (2).Re{s} = a (3). Re{s} > -a (4). Re{s} < -a	4
9.	The Laplace transform of x(at) is __________ (1). $\left\|\frac{1}{a}\right\|X\left\|\frac{S}{a}\right\|$ (2). $X\left\|\frac{S}{a}\right\|$ (3). $\left\|\frac{1}{a}\right\|X(S)$ (4). None of the above	1
10.	The Laplace transform of $x(t)e^{S_0 t}$ is __________ (1). $\left\|\frac{1}{a}\right\|X\left\|\frac{S}{a}\right\|$ (2). $X(S-S_0)$ (3). $X(S+S_0)$ (4). None of the above	2
11.	The Laplace transform of $\frac{d^n}{dt^n}x(t)$ is __________ (1). $X(s)$ (2). $SX(S)$ (3). $S^n X(S)$ (4). None of the above	3

12.	The Laplace transform of the x(t) = u(t) – u(t-T) (1). $\frac{1+e^{-s}}{s}$ (2). $\frac{e^{-s}}{s}$ (3). $\frac{1}{s}$ (4). $\frac{1-e^{-s}}{s}$	4
13.	The Laplace transform of x(t) is X(S) = $\frac{2(S+1)}{S^2+2S+5}$, then the initial value of x(t) is_ (1). 2 (2). 0 (3). -2 (4). 5	1
14.	The Laplace transform of x(t) is X(S) = $\frac{2(S+1)}{S^2+2S+5}$, then the final value of x(t) is__ (1). 2 (2). 0 (3). -2 (4). 5	2
15.	The ROC of the Laplace transform of the function $x(t)=e^{-(a+2)t+5}u(t)$ is________ (1). Re{s} > a+7 (2). Re{s} > a+5 (3). Re{s} > a+2 (4). Re{s} > a	3
16.	If x(t) is two sided signal, then its ROC is_________ (1). Right sided (2). Left sided (3). Entire S - Plane (4). Finite duration strip, which lies between two poles.	4
17.	If x(t) is right sided signal, then its ROC is_________ (1). Right half of the S-Plane (2). Left half of the S-Plane (3). Entire S - Plane (4). Finite duration strip, which lies between two poles.	1
18.	The Laplace transform of Cos(at)u(t) is______ (1). $\frac{a}{S^2+a^2}$ (2). $\frac{S}{S^2+a^2}$ (3). $\frac{a}{S^2-a^2}$ (4). $\frac{S}{S^2-a^2}$	2
19.	The Laplace transform of $e^{-at}Cos(bt)$ is______ (1). $\frac{s-b}{(S+b)^2+a^2}$ (2). $\frac{S}{(S+b)^2+a^2}$ (3). $\frac{s+b}{(S+b)^2+a^2}$ (4). $\frac{a}{(S+b)^2+a^2}$	3
20.	The inverse Laplace transform of $X(s)=\frac{s+5}{(s+1)(s+3)}$ (1). $e^{-t}-e^{-3t}$ (2). $e^{-3t}+e^{-t}$ (3). $2e^{-t}+e^{-3t}$ (4). $2e^{-t}-e^{-3t}$	4
21.	If X(s) is Laplace transform of x(t), then the Laplace transform of $\int_{-\infty}^{t} x(\tau)d\tau$ is____ (1). $\frac{X(S)}{S}$ (2). $SX(S)-x(0)$ (3). $SX(S)$ (4). None of the above	1
22.	Laplace transforms of f(t) and g(t) are F(s) and G(s), respectively. Which one of the following expressions gives the inverse Laplace transform of F(s)G(s)? (1). f(t)g(t) (2). f(t)*g(t) (3). f(t)-g(t) (4). None of the above	2
23.	The Laplace transform of x(t) is X(S) = $\frac{(S+3)}{S(S+2)}$, then the initial value of x(t) is____ (1). 3/2 (2). 0 (3). -2 (4). 1	4
24.	The Laplace transform of x(t) is X(S) = $\frac{(S+3)}{S(S+2)}$, then the final value of x(t) is____ (1). 3/2 (2). 0 (3). -2 (4). 1	1
25.	A system is said to be causal, if it's all poles of system function ______ (1). Lies on right side of the ROC. (2). Lies on left side of the ROC. (3). Includes jw axis (4). None of the above	2

26.	A system is said to be stable, if it's all poles of system function _____ (1). Lies on right side of the ROC. (2). Lies on left side of the ROC. (3). Includes jw axis (4). None of the above	3
27.	The inverse Laplace transform of $X(S)=\frac{e^{-as}}{s}$ is_____ (1). e^{-at} (2). $u(t)$ (3). $(t-a)u(t-a))$ (4). $u(t-a)$	4
28.	The inverse Laplace transform of $X(S)=\frac{2}{(S+4)(S-1)}$, if ROC re{s} > 1 (1). $-\frac{2}{5}e^{-4t}u(t)+\frac{2}{5}e^{t}u(t)$ (2). $\frac{2}{5}e^{-4t}u(t)-\frac{2}{5}e^{t}u(t)$ (3). $-\frac{2}{5}e^{-4t}u(t)-\frac{2}{5}e^{t}u(t)$ (4). None of the above	1
29.	If the poles are lies to left sided of ROC, its associated time domain signal is_____ 1). Left sided signal. (2). Right sided signal. (3). Two sided (4). None of the above	2

30.	What is the ROC of ZT[u(n)] (A)|z|>0 (B)|z|>1 (C)0<|z|<1 (D)|z)<0	B
31.	Find the z-domain of x(n)={1,0,1} (A)$1+z^{-1}$ (B) $z+1+z^{-1}$ (C) $z+z^{-1}$ (D) 1+z	A
32.	Determine the Z-transform of x(n)=δ(n-9)	z^{-9}
33.	What is the ROC of z transform of " $(\frac{1}{2})^n u(n) + (\frac{1}{3})^n u(n) + u(n)$ "	|z|>1
34.	If ZT of $a^n u(n) * b^n u(n)$ is X(z)/(z-a)(z-b), then X(z) is (A)1 (B) z (C) z^2 (D) 1+z	C
35.	If ZTof u(-n) is k/X(z), then k-X(z) is	z
36.	Find a causal sequence from the z-domain $X(z) = \frac{1}{z-1}$	u(n-1)
37.	If $X(z) = \frac{9}{z^9(z-9)}$ and $x(n)=a^{n-b}u(n-c)$, then a,b,c= (A)9,9,9 (B) 9,10,9 (C) 10,9,9 (D) 9,9,10	D
38.	Find the initial value of a sequence x(n) from $X(z) = \frac{z(z+1)}{(3z-1)(2z-1)}$	1/6
39.	Find the final value of a sequence x(n) from $X(z) = \frac{z(z+1)}{(z-1)(2z-1)}$	2